Homework 2: IERG 6300

Due date: Feb 5, 2022.

Exercises

1. Show that if every sequence of measurable functions f_n that converges to zero almost everywhere, also converges in expectation with respect to a finitely additive measure to 0, then the measure is also countably additive.

2. Let T be a measurable map from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$, and P be a probability measure on $(\Omega_1, \mathcal{F}_1)$. For any $A \in \mathcal{F}_2$ define

$$Q(A) = P(T^{-1}(A)).$$

   Verify that Q is a probability measure on $(\Omega_2, \mathcal{F}_2)$.

3. (Markov's inequality) Let f be a non-negative random variable defined on a underlying probability space $(\Omega, \mathcal{F}, P)$. For $A \in \mathcal{F}$ let $m_A = \inf_{\omega \in A} f(\omega)$. Then, show that,

$$m_A P(A) \leq \mathrm{E}(f 1_A) \leq \mathrm{E}(f).$$

4. Let $p, q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Show that for any $x, y \geq 0$

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

   Now let X and Y be two non-negative random variables such that $\mathrm{E}(X^p)$ and $\mathrm{E}(Y^q)$ are positive and finite. Now prove Hölder's inequality that

$$\mathrm{E}(XY) \leq \mathrm{E}(X^p)^{\frac{1}{p}} \, \mathrm{E}(Y^q)^{\frac{1}{q}}.$$

5. Let Ω, Ω' be two spaces. Let $\mathcal{B}$ be a collection of subsets of Ω'. Let $f : \Omega \to \Omega'$ be a mapping. Let $\mathcal{E}$ be a collection of subsets of Ω obtained as follows: $\mathcal{E} = \{E : E \subseteq \Omega, E = f^{-1}(B), B \in \mathcal{B}\}$. Show that f is a measurable mapping from: $(\Omega, \sigma(\mathcal{E})) \mapsto (\Omega', \sigma(\mathcal{B}))$. Here $\sigma(\mathcal{A})$

refers to the σ-field generated by $\mathcal{A}$. (Essentially you have to show that for every $C \in \sigma(\mathcal{B})$ one has $f^{-1}(C) \in \sigma(\mathcal{E})$. In particular, this result implies that to check measurability of mappings, one just need to consider inverse images of any collection of sets that generate the σ-field.)

6. Show that if P is a probability measure on $(\mathbb{R}, \mathcal{B})$ then for every $A \in \mathcal{B}$ and $\epsilon > 0$ there exists an open set G containing A such that $\mathrm{P}(G) < \mathrm{P}(A) + \epsilon$. Such a probability measure is a *regular probability measure*. Here $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

7. Show that any monotone function $g : \mathbb{R} \mapsto \mathbb{R}$ is Borel-measurable.