On Gaussian Extremizers for the Capacity Region of the Gaussian Interference Channel

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The Gaussian interference channel models the situation of multiple mutually interfering point-to-point communications over a shared medium with additive Gaussian noise. For such a setting, a quantity of interest is the set of message-rates at which one can communicate reliably, or also called the capacity region. In this thesis we focus exclusively on the two sender-receiver pair case. The primary research questions tackled in this thesis concern the evaluation of and determining the optimality of the Han–Kobayashi achievable region for this setting. A key difficulty arises from the non-convex nature of the optimization problem associated with computing the Han–Kobayashi achievable region, even restricted to Gaussian input distributions.

In this thesis, we show that the multi-letter extensions of the Han–Kobayashi achievable region with Gaussian input distributions do not improve on the single-letter case. A second contribution of this thesis is the formulation of a conjecture concerning the Gaussian extremality of a functional, that would imply that the Han–Kobayashi achievable region with Gaussian input distributions will match the capacity region for the Gaussian Z-interference channel. Finally, we establish the above conjecture for some parameters which then implies the optimality of the Han–Kobayashi achievable region for a collection of weighted sum-rates for the Gaussian Z-interference channel.

摘要

《論高斯干擾信道容量的高斯極值》

高斯干擾信道為一描述帶加性高斯雜訊的共用傳輸介質上互相干擾的多個點對點通信的模型。此模型一重要的量為能使信息可靠傳送的信息率之集合，又稱容量區域。本論文專注於兩對發送端和接收端的情況。本論文研究的主要問題為高斯干擾信道的韓–小林（Han–Kobayashi）可達區域的計算和最優性判定。此問題的關鍵難處之一在於，即使限制輸入為高斯分布，與計算韓–小林可達區域相關的最佳化問題仍非凸。

我們在本論文中證明以高斯分布為輸入的韓–小林可達區域的多字符擴展並不能改進單字符區域。其次，本論文提出一關於一泛函高斯極值的猜想；此猜想若成立，則對於高斯Z干擾信道而言，以高斯分布為輸入的韓–小林可達區域等於信道容量區域。最後，我們證明此猜想的部分情況，並由此推論高斯Z干擾信道的韓–小林可達區域極大化若干加權總和速率。

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Notations

Here we describe the notations used in this thesis.

- $A := B$ means A is defined as B.
- $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^n$ is the n-dimensional Euclidean space ($n \geq 1$). Elements of $\mathbb{R}^n$ are column vectors.
- $\|x\|$ denotes the Euclidean norm of a real vector x.
- $\langle x, y\rangle$ denotes the Euclidean inner product of real vectors x and y.
- A^T denotes the transpose of a matrix A, $\mathrm{tr}(A)$ denotes its trace, and $|A|$ denotes its determinant. $A \succeq 0$ means A is positive semidefinite. $A \succeq B$ means $A - B \succeq 0$.
- The cardinality of a set S is denoted by $|S|$.
- Random variables in general are denoted by uppercase letters (e.g. $X, Y, Z, \dots$) and their realizations by lowercase letters (e.g. $x, y, z, \dots$). The sets in which random variables take values are denoted by uppercase calligraphic letters (e.g. $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \dots$).
- Vector random variables are denoted by uppercase boldface letters (e.g. $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \dots$) and their realizations by lowercase boldface letters (e.g. $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$).
- X_i^j denotes the tuple $(X_i, X_{i+1}, \dots, X_j)$ for $i \leq j$. We may omit the subscript when it is 1, i.e., $X^j = (X_1, X_2, \dots, X_j)$. When it is clear from the context, we may denote the tuple $(X_{i1}, X_{i2}, \dots, X_{in})$ by X_i^n.
- $p(y|x)$ is a collection of distributions on $\mathcal{Y}$, one for every $x \in \mathcal{X}$.
- $X \sim p(x)$ means X follows the distribution $p(x)$.
- $X \sim \mathcal{N}(\mu, K)$ means X is a Gaussian random variable with mean μ and covariance matrix (or variance) K.

- $X_1 \to X_2 \to \cdots \to X_n$ ($n \geq 3$) forms a Markov chain if $p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2|x_1)\ldots p(x_n|x_{n-1})$.

- $h(X)$ is the differential entropy of X. $h(X|Y)$ is the conditional differential entropy of X given Y.

- $I(X;Y)$ is the mutual information of X and Y. $I(X;Y|Z)$ is the conditional mutual information of X and Y given Z.

- $\mathrm{P}(E)$ is the probability of event E.

- $\mathrm{E}[X]$ is the expectation (also called mean) of X. $\mathrm{E}[X|Y]$ is the conditional expectation of X given Y.

- $\mathrm{Cov}(\mathbf{X}) := \mathrm{E}[(\mathbf{X} - \mathrm{E}[\mathbf{X}])(\mathbf{X} - \mathrm{E}[\mathbf{X}])^T]$ is the covariance matrix of $\mathbf{X}$. $\mathrm{Cov}(\mathbf{X}|U)$ is the conditional covariance matrix of $\mathbf{X}$ given U, defined as $\mathrm{Cov}(\mathbf{X}|U) := \mathrm{E}[(\mathbf{X} - \mathrm{E}[\mathbf{X}|U])(\mathbf{X} - \mathrm{E}[\mathbf{X}|U])^T|U]$.

- $\mathcal{C}_x[f(x)]$ is the upper concave envelope of the functional f, defined as in Definition 4.1.

- $\mathcal{R}_n^{\mathrm{HK}}$ is the n-letter extension of the Han–Kobayashi achievable region for an interference channel, defined as in Definition 1.2.

- $\mathcal{R}^{\text{HK-GS}}$ is the Han–Kobayashi achievable region with Gaussian inputs for a Gaussian interference channel, defined as in Definition 1.3. $\mathcal{R}_n^{\text{HK-GS}}$ is its n-letter extension, defined as in Definition 1.4.

Chapter 1

Introduction

As a natural extension to Shannon's groundbreaking theory of the quantitative treatment of point-to-point communication [Sha48], the field of network information theory [EGK11] studies the fundamental limits on information flow in multi-user communication networks. Classical mathematical models of fundamental multi-terminal communication settings including the multiple-access channel, the broadcast channel and the interference channel have been proposed as early as in the 70s. Thanks to technological advancements, in particular the advent of wireless communications, interest in the field boomed as the demand for high-throughput wireless communications soared in the mid-90s. Network information theory often deals with the capacities of network models abstracted from practical scenarios as well as practical coding schemes that approach the fundamental limits. Apart from mathematical curiosity, understanding the structures behind these limits has given insights on implementing efficient coding schemes in real-world networks.

There is not yet a general theory: While the point-to-point communication setting has been well-studied since the time of Shannon, most of the fundamental problems in multi-user communications settings nevertheless remain unsolved. This thesis focuses on the Gaussian interference channel, which is a model commonly encountered in wireless communications. This abstracts a communication setting where multiple point-to-point links suffer from crosstalk from each other and noise from the environment. This thesis focuses exclusively on the two sender-receiver pair case. The study of this setting dates back to the mid-70s, however there is yet a computable characterization of the set of data rates that allows reliable communications, or also called the capacity region, for the general case. Nevertheless, there is a promising candidate for the capacity region, namely, the Han–Kobayashi achievable region with Gaussian inputs, whose optimality, if shown, would solve the long-standing fundamental open problem. This thesis investigates the optimality of the Han–Kobayashi achievable region with Gaussian inputs for the two-sender-receiver-pair Gaussian interference channel. The results in this thesis provide evidence suggesting the optimality of the Han–Kobayashi achievable region with Gaussian inputs. For the one-sided interference setting this thesis conjectures an information inequality, whose veracity would imply the optimality of the Han–Kobayashi achievable region with Gaussian inputs for this setting. Furthermore, this thesis establishes the above conjecture in certain non-trivial parameter regimes which also provide new results regarding the capacity region for the above setting.

1.1 Interference channel

The **interference channel** as first introduced by Ahlswede [Ahl74] is the classical model for the scenario of two mutually interfering point-to-point communications over a shared medium. A discrete memoryless interference channel consists of two sets of input alphabets $\mathcal{X}_1, \mathcal{X}_2$, two sets of output alphabets $\mathcal{Y}_1, \mathcal{Y}_2$, and a stochastic map $p(y_1, y_2|x_1, x_2)$ from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{Y}_1 \times \mathcal{Y}_2$.

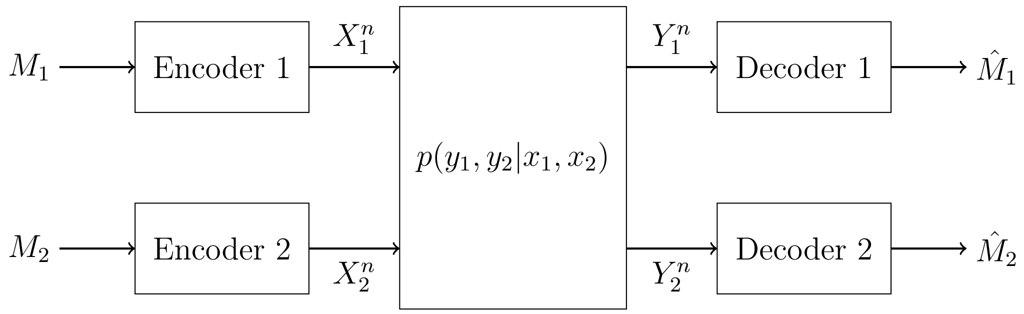


Figure 1.1: A discrete memoryless interference channel

As shown in Figure 1.1, in a communication scenario, a transmitter wishes to send a message M_1 to its receiver, whilst another transmitter wishes to send a message M_2, independent of M_1, to its receiver, and both communications use a shared medium. Both transmitters respectively encode their messages into the codewords $X_1^n \in \mathcal{X}_1^n$ and $X_2^n \in \mathcal{X}_2^n$ which are then sent over the channel, and the receivers respectively decode the received vectors $Y_1^n \in \mathcal{Y}_1^n$ and $Y_2^n \in \mathcal{Y}_2^n$ they see at the receiving sides to recover the messages.

We adopt the standard definition of achievable rates and capacity region. The reader is referred to [EGK11] for the standard notions of network information theory. A $(2^{nR_1}, 2^{nR_2}, n)$ **code** for the interference channel consists of two **message sets** $\mathcal{M}_i := \{1, 2, \dots, \lfloor 2^{nR_i} \rfloor\}$ (where $\lfloor x \rfloor$ denotes the largest integer $\leq x$), two **encoding functions** $\mathrm{Enc}_i : \mathcal{M}_i \to \mathcal{X}_i^n$ and two **decoding func-**

tions $\mathrm{Dec}_i : \mathcal{Y}_i^n \to \mathcal{M}_i$, where $i = 1, 2$. For a $(2^{nR_1}, 2^{nR_2}, n)$ code, the **average probability of error** is defined by

$$P_e^{(n)} := \mathrm{P}(M_1 \neq \hat{M}_1 \text{ or } M_2 \neq \hat{M}_2),$$

where the pair of transmitted messages (M_1, M_2) is assumed to be uniformly distributed over $\mathcal{M}_1 \times \mathcal{M}_2$, $X_i^n := \mathrm{Enc}_i(M_i)$ is the codeword sent by transmitter i, and $\hat{M}_i := \mathrm{Dec}_i(Y_i^n)$ is the decoded message at receiver i $(i = 1, 2)$, where the received vectors Y_1^n, Y_2^n follow the distribution

$$(Y_1^n, Y_2^n) \sim \prod_{j=1}^n p(y_{1j}, y_{2j} | x_{1j}, x_{2j}).$$

A rate pair $(R_1, R_2) \in \mathbb{R}_{\geq 0}^2$ is **achievable** if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes whose average probability of error goes to 0 as n goes to infinity. A subset of $\mathbb{R}_{\geq 0}^2$ is an **achievable region** if it is the closure of a set consisting of achievable rate pairs. The **capacity region** is defined to be the closure of the set of all achievable rate pairs.

1.1.1 Han–Kobayashi achievable region

The best achievable region known for a general discrete memoryless interference channel is given by Han and Kobayashi [HK81]. However it has been shown [NXY15] that there is some discrete memoryless interference channel for which the Han–Kobayashi achievable region is strictly contained in the capacity region, which was shown by establishing the strict inclusion of single-letter Han–Kobayashi achievable region inside its multi-letter extension. It is known that the limit of multi-letter extension of the Han–Kobayashi achievable region is the capacity region, as shown in Proposition 1.1.

Definition 1.1. *The **Han–Kobayashi achievable region** of a discrete memoryless interference channel $p(y_1, y_2|x_1, x_2)$ is the set of rate pairs (R_1, R_2) such that*

$$R_1 \leq I(X_1; Y_1 | U_2, Q), \tag{1.1a}$$

$$R_2 \leq I(X_2; Y_2 | U_1, Q), \tag{1.1b}$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q), \tag{1.1c}$$

$$R_1 + R_2 \leq I(U_1, X_2; Y_2 | Q) + I(X_1; Y_1 | U_1, U_2, Q), \tag{1.1d}$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1 | U_1, Q) + I(U_1, X_2; Y_2 | U_2, Q), \tag{1.1e}$$

$$2R_1 + R_2 \leq I(U_2, X_1; Y_1 | Q) + I(X_1; Y_1 | U_1, U_2, Q) + I(U_1, X_2; Y_2 | U_2, Q), \tag{1.1f}$$

$$R_1 + 2R_2 \leq I(U_1, X_2; Y_2 | Q) + I(X_2; Y_2 | U_1, U_2, Q) + I(U_2, X_1; Y_1 | U_1, Q) \tag{1.1g}$$

for some $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$.

Theorem 1.1 ([HK81], Theorem 6.4 of [EGK11])**.** *The Han–Kobayashi achievable region of a discrete memoryless interference channel, defined as in Definition 1.1, is an achievable region.*

Definition 1.2. *Let $n \geq 1$. The* ***n-letter extension of the Han–Kobayashi achievable region****, denoted by $\mathcal{R}_n^{HK}$, for an interference channel is the set of rate pairs (R_1, R_2) such that (nR_1, nR_2) is in the Han–Kobayashi achievable region of the interference channel obtained by taking n independent copies of the original interference channel.*

Proposition 1.1. *For an interference channel, the closure of $\bigcup_{n=1}^{\infty} \mathcal{R}_n^{HK}$ is the capacity region.*

Proof. One can see by grouping channel uses into blocks of n time slots that $\mathcal{R}_n^{\mathrm{HK}}$ yields an achievable region for the original interference channel for every $n \geq 1$. So it suffices to show that the capacity region is contained inside the closure of $\bigcup_{n=1}^{\infty} \mathcal{R}_n^{\mathrm{HK}}$. A standard application of Fano's inequality gives that for any sequence of codebooks of rate (R_1, R_2), whose average probability of error goes to zero, there exists two sequences ϵ_n, ϵ'_n of nonnegative real numbers such that $\epsilon_n, \epsilon'_n \to 0$ as $n \to \infty$ and

$$\begin{aligned} R_1 - \epsilon_n &\leq \frac{1}{n} I(X_1^n; Y_1^n), \\ R_2 - \epsilon'_n &\leq \frac{1}{n} I(X_2^n; Y_2^n), \end{aligned}$$

which implies that $(R_1 - \epsilon_n, R_2 - \epsilon'_n) \in \mathcal{R}_n^{\mathrm{HK}}$ which can be seen by setting U_1, U_2, Q to be constants in the equations (1.1). Hence every achievable rate pair is a limit of sequence in $\bigcup_{n=1}^{\infty} \mathcal{R}_n^{\mathrm{HK}}$. This completes the proof. □

A special case of interest for the interference channel is the **Z-interference channel**, which is when only one of the two transmitter-receiver pairs suffer from crosstalk from the other transmitter, or, without loss of generality, Y_1 is independent of X_2. For a Z-interference channel, the Han–Kobayashi achievable region reduces to the following.

Proposition 1.2. *The Han–Kobayashi achievable region of a discrete memoryless Z-interference channel $p(y_1|x_1)p(y_2|x_1, x_2)$ is equal to the set of all rate pairs (R_1, R_2) such that*

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|Q), \\ R_2 &\leq I(X_2; Y_2|U_1, Q), \\ R_1 + R_2 &\leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q) \end{aligned}$$

for some $p(q)p(u_1, x_1|q)p(x_2|q)$.

1.2 Gaussian interference channel

The **Gaussian interference channel** (GIC) as shown in Figure 1.2 is an instance of an interference channel where the channel has additive Gaussian noise. The general setting of a GIC is given by

$$
\begin{aligned}
Y_1 &= X_1 + bX_2 + Z_1, \\
Y_2 &= X_2 + aX_1 + Z_2,
\end{aligned}
$$

where $a, b \geq 0$ are real constants, $X_i, Y_i \in \mathbb{R}$ are the transmitted symbol at transmitter i and the received symbol at receiver i ($i = 1, 2$), respectively, and $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ are standard scalar Gaussian random variables such that X_1, X_2, Z_1, Z_2 are mutually independent. We assume the codebooks X_1^n, X_2^n satisfy the expected average power constraints

$$
\begin{aligned}
\mathrm{E}[\|X_1^n\|^2] &\leq nP_1, \\
\mathrm{E}[\|X_2^n\|^2] &\leq nP_2,
\end{aligned}
$$

where $P_1, P_2 \geq 0$. The **Gaussian Z-interference channel** (GZIC) is the special case of GIC where $a > 0$ and $b = 0$.

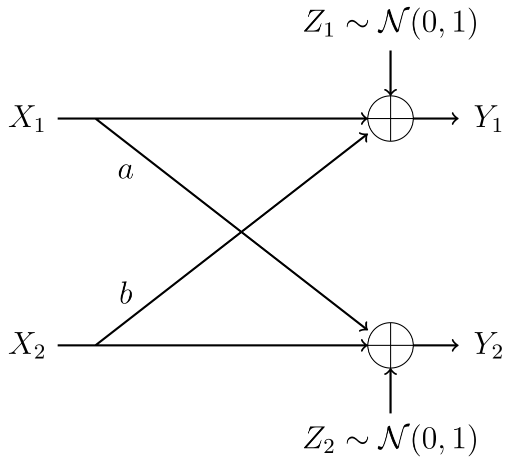


Figure 1.2: Gaussian interference channel

The GIC has been actively studied since mid-70s. Carleial [Car75] first introduced the GIC with expected average power constraints and showed that in the regime of very strong interference ($a \geq \sqrt{1 + P_2}$ and $b \geq \sqrt{1 + P_1}$) the capacity region of GIC is the same as that without the interference. Sato [Sat81] generalized Carleial's result by determining the capacity region of GIC with strong interference to be the region requiring both transmitted messages to be decoded at both receivers, and similarly for strong Z-interference. The best achievable region known for GIC is the Han–Kobayashi achievable region [HK81]. Determining a computable characterization of the capacity region of GIC for general range of parameters has been a long-standing central open problem in the field.

1.2.1 Known results for capacity region of GIC

We summarize some known results for the capacity region of GIC.

(i) For **strong interference** $(a, b \geq 1)$ the capacity region is known [Sat81] to be the set of rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1+P_1), \\ R_2 &\leq \frac{1}{2}\log(1+P_2), \\ R_1 + R_2 &\leq \min\left\{\frac{1}{2}\log(1+a^2P_1+P_2), \frac{1}{2}\log(1+P_1+b^2P_2)\right\}. \end{aligned}$$

This region is the same as requiring both transmitted messages to be decoded at both receivers.

(ii) For **strong Z-interference** ($a \geq 1$ and $b = 0$) the capacity region is known [Sat81] to be the set of rate pairs (R_1, R_2) such that

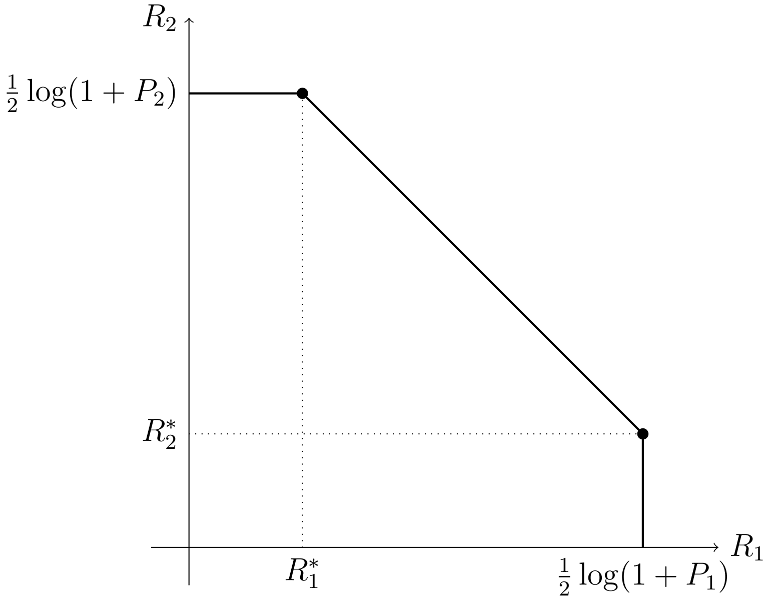
$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1+P_1), \\ R_2 &\leq \frac{1}{2}\log(1+P_2), \\ R_1 + R_2 &\leq \frac{1}{2}\log(1+a^2P_1+P_2). \end{aligned}$$

This region is the same as requiring receiver 2 to decode both transmitted messages.

(iii) For **weak Z-interference** ($0 < a \leq 1$ and $b = 0$) the capacity region is the same as mixed interference with $b = \frac{1}{a}$ and the same a [Cos85a].

(iv) The capacity region has two extreme points, also called **corner points**, respectively of the form (C_1, R_2^*) and (R_1^*, C_2), where $C_i := \frac{1}{2}\log(1+P_i)$ $(i = 1, 2)$ is the interference-free point-to-point capacity and R_1^*, R_2^* have been determined for all ranges of parameters.

In the cases of **mixed interference** ($0 < a \leq 1$ and $b \geq 1$), **weak interference** ($0 < a \leq 1$ and $0 < b \leq 1$) and **weak Z-interference** ($0 < a \leq 1$ and $b = 0$) the corner points are:



$$R_1^* = \begin{cases} \frac{1}{2}\log\left(1+\frac{a^2P_1}{1+P_2}\right) & \text{if } b \geq \sqrt{1+\frac{P_1}{P_2}(a^2-1)} \text{ or } b=0, \\ \frac{1}{2}\log\left(1+\frac{P_1+(b^2-1)P_2}{1+P_2}\right) & \text{otherwise.} \end{cases}$$

$$R_2^* = \begin{cases} \frac{1}{2}\log\left(1+\frac{(a^2-1)P_1+P_2}{1+P_1}\right) & \text{if } b \geq \sqrt{1+\frac{P_1}{P_2}(a^2-1)} \text{ or } b=0, \\ \frac{1}{2}\log\left(1+\frac{b^2P_2}{1+P_1}\right) & \text{otherwise.} \end{cases}$$

Figure 1.3: Capacity region of GIC when $a \geq 1$ and ($b \geq 1$ or $b = 0$)

- **Costa–Sato corner point** [Sat81], [Cos85b]:

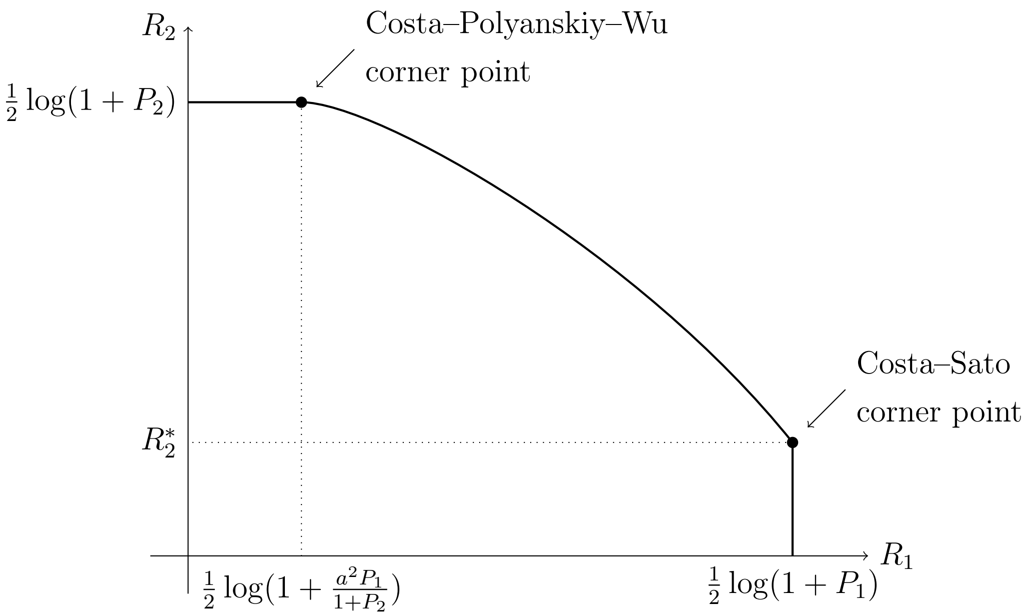
$$(R_1, R_2) = \begin{cases} \left(C_1, \frac{1}{2}\log\left(1+\frac{P_2}{1+a^2P_1}\right)\right) & \text{if } b \geq \sqrt{\frac{1+P_1}{1+a^2P_1}} \text{ or } b=0, \\ \left(C_1, \frac{1}{2}\log\left(1+\frac{b^2P_2}{1+P_1}\right)\right) & \text{otherwise.} \end{cases}$$

- **Costa–Polyanskiy–Wu corner point** [Cos85b], [PW15]:

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{a^2P_1}{1+P_2}\right), C_2\right).$$

Note that in the case of weak interference these two corner points are essentially the same due to symmetry, and in such case both corner points are called **Costa–Polyanskiy–Wu corner point**.

These corner points have a rich history in the field. The extremality of the former corner point for GIC with mixed interference is established by Costa [Cos85a] as a consequence of a work of Sato [Sat81]. Costa [Cos85a] first claimed the extremality of the latter corner point via concavity of entropy power. However Sason [Sas04] observed a gap (Lemma 1 of [Cos85b]) in the finishing part of the proof. Polyanskiy and Wu [PW15] fixed the proof of Lemma 1 of [Cos85b] using Talagrand's HWI inequality [Tal96] and thus established the extremality of the latter corner point.



$$R_2^* = \begin{cases} \frac{1}{2}\log\left(1+\frac{P_2}{1+a^2P_1}\right) & \text{if } b \geq \sqrt{\frac{1+P_1}{1+a^2P_1}} \text{ or } b=0, \\ \frac{1}{2}\log\left(1+\frac{b^2P_2}{1+P_1}\right) & \text{otherwise.} \end{cases}$$

Figure 1.4: Capacity region of GIC when $0 < a \leq 1$ and $b \geq 0$

(v) For weak interference with $a(1+b^2P_2)+b(1+a^2P_1) \leq 1$, the treating-interference-as-noise point

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{P_1}{1+b^2P_2}\right), \frac{1}{2}\log\left(1+\frac{P_2}{1+a^2P_1}\right)\right)$$

achieves the maximum sum-rate R_1+R_2 [AV09; MK09; SKC09]. Moreover, there is a discontinuity of the slope of boundary of capacity region at this point, as shown by [Bei+16] for the symmetric interference case, and the extension of this result to the general case is given as Theorem 3.4.

1.2.2 Han–Kobayashi achievable region for GIC

For GIC, the Han–Kobayashi achievable region coincides with the capacity region in the regimes of strong interference and strong Z-interference whilst whether the regions coincide in general is unknown.

We consider the Han–Kobayashi achievable region with input distribution restricted to be Gaussians. It is an open question whether the Han–Kobayashi achievable region evaluated with Gaussian inputs, as defined in Definition 1.3,

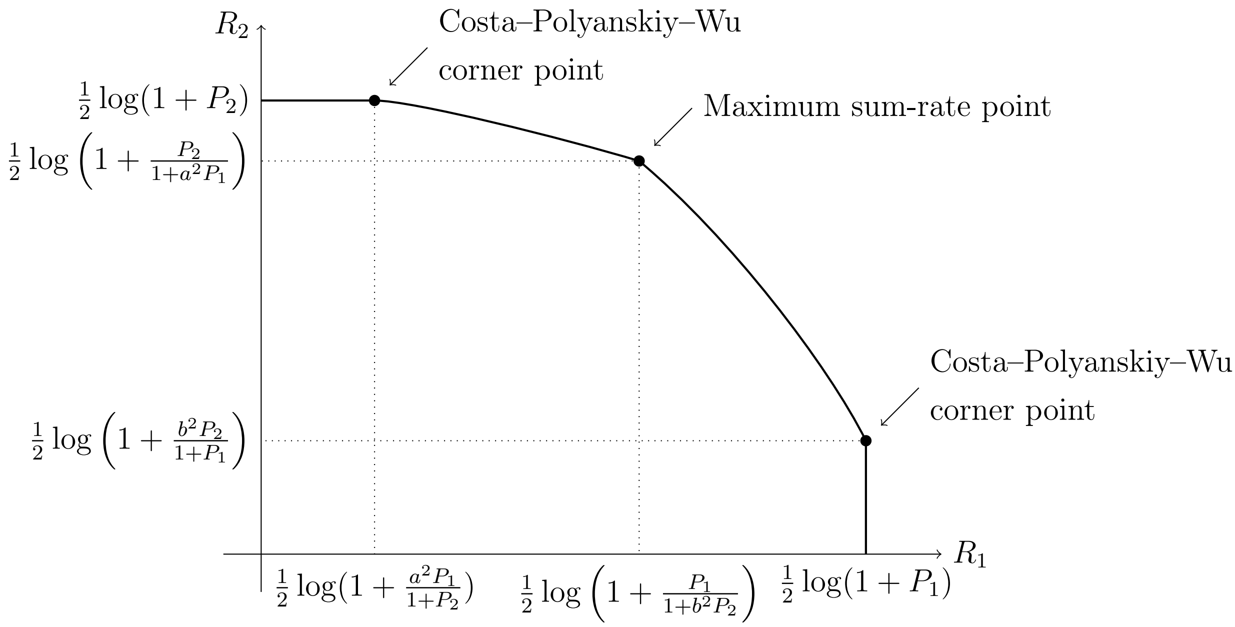


Figure 1.5: Capacity region of GIC when $a(1+b^2P_2)+b(1+a^2P_1)\leq 1$, showing slope discontinuity at the maximum sum-rate point

matches the capacity region. It is known [ETW08] that the Hausdorff distance (under L^1-norm) between the capacity region and the Han–Kobayashi achievable region with Gaussian inputs is at most 1, for all ranges of parameters.

Definition 1.3. *The* ***Han–Kobayashi achievable region with Gaussian inputs for GIC****, denoted by* $\mathcal{R}^{\textit{HK-GS}}$*, where GS stands for Gaussian signaling, is the set of rate pairs* (R_1, R_2) *such that the inequalities* (1.1) *with* $X_i := U_i + V_i$ $(i = 1, 2)$ *hold for some* $p(q)p(u_1|q)p(v_1|q)p(u_2|q)p(v_2|q)$ *such that the conditional distributions* $p(u_1|q), p(v_1|q), p(u_2|q), p(v_2|q)$ *are zero-mean Gaussian distributions for each* q*, and that the power constraints*

$$\mathrm{E}\left[U_1^2+V_1^2\right]\leq P_1,$$
$$\mathrm{E}\left[U_2^2+V_2^2\right]\leq P_2$$

are satisfied.

As an attempt to disprove optimality, motivated by the result in [NXY15] in which its authors establish the strict sub-optimality of Han–Kobayashi achievable region for a certain interference channel by showing that the two-letter extension of the Han–Kobayashi achievable region strictly improves on the single-letter region, we naturally ask whether the same phenomenon happens for the Han–Kobayashi achievable region with Gaussian inputs for GIC, for, if so, would constitute a proof for the strict sub-optimality of this scheme. In Chapter 2 of this thesis we will answer this question in the negative.

Definition 1.4. *Let $n \geq 1$. The n-**letter extension of the Han–Kobayashi achievable region with Gaussian inputs for GIC**, denoted by $\mathcal{R}_n^{HK\text{-}GS}$, where GS stands for Gaussian signaling, refers to the set of rate pairs (R_1, R_2) satisfying*

$$\begin{aligned}
R_1 &\leq \frac{1}{n} I(\mathbf{X}_1; \mathbf{Y}_1 | \mathbf{U}_2, Q), \\
R_2 &\leq \frac{1}{n} I(\mathbf{X}_2; \mathbf{Y}_2 | \mathbf{U}_1, Q), \\
R_1 + R_2 &\leq \frac{1}{n} \left[I(\mathbf{U}_2, \mathbf{X}_1; \mathbf{Y}_1 | Q) + I(\mathbf{X}_2; \mathbf{Y}_2 | \mathbf{U}_1, \mathbf{U}_2, Q) \right], \\
R_1 + R_2 &\leq \frac{1}{n} \left[I(\mathbf{U}_1, \mathbf{X}_2; \mathbf{Y}_2 | Q) + I(\mathbf{X}_1; \mathbf{Y}_1 | \mathbf{U}_1, \mathbf{U}_2, Q) \right], \\
R_1 + R_2 &\leq \frac{1}{n} \left[I(\mathbf{U}_2, \mathbf{X}_1; \mathbf{Y}_1 | \mathbf{U}_1, Q) + I(\mathbf{U}_1, \mathbf{X}_2; \mathbf{Y}_2 | \mathbf{U}_2, Q) \right], \\
2R_1 + R_2 &\leq \frac{1}{n} \left[I(\mathbf{U}_2, \mathbf{X}_1; \mathbf{Y}_1 | Q) + I(\mathbf{X}_1; \mathbf{Y}_1 | \mathbf{U}_1, \mathbf{U}_2, Q) + I(\mathbf{U}_1, \mathbf{X}_2; \mathbf{Y}_2 | \mathbf{U}_2, Q) \right], \\
R_1 + 2R_2 &\leq \frac{1}{n} \left[I(\mathbf{U}_1, \mathbf{X}_2; \mathbf{Y}_2 | Q) + I(\mathbf{X}_2; \mathbf{Y}_2 | \mathbf{U}_1, \mathbf{U}_2, Q) + I(\mathbf{U}_2, \mathbf{X}_1; \mathbf{Y}_1 | \mathbf{U}_1, Q) \right],
\end{aligned}$$

where $\mathbf{X}_i := \mathbf{U}_i + \mathbf{V}_i$ $(i = 1, 2)$, for some $p(q)p(\mathbf{u}_1|q)p(\mathbf{v}_1|q)p(\mathbf{u}_2|q)p(\mathbf{v}_2|q)$ such that the conditional distributions $p(\mathbf{u}_1|q), p(\mathbf{v}_1|q), p(\mathbf{u}_2|q), p(\mathbf{v}_2|q)$ are n-dimensional zero-mean Gaussian distributions for each q, and that the power constraints

$$\begin{aligned}
\mathrm{E}_Q \left[\mathrm{tr} \left(\mathrm{Cov}(\mathbf{U}_1|Q) + \mathrm{Cov}(\mathbf{V}_1|Q) \right) \right] &\leq nP_1, \\
\mathrm{E}_Q \left[\mathrm{tr} \left(\mathrm{Cov}(\mathbf{U}_2|Q) + \mathrm{Cov}(\mathbf{V}_2|Q) \right) \right] &\leq nP_2
\end{aligned}$$

are satisfied.

On the other hand, a major obstacle hindering proving the optimality of the Han–Kobayashi achievable region with Gaussian inputs is that time-sharing between Gaussian distributions, or commonly called power control in the literature, is known to strictly improve on the achievable region with the time-sharing variable Q being constant, as shown in [Cos11; CN12]. A naive application of the previous techniques, either the monotone path argument [Sta59] or the rotation/doubling based argument [GN14], will only work if time-sharing did not improve on the achievable region. Motivated by this observation, in Chapter 4 we propose a conjecture based on the Fenchel dual representation of the upper concave envelope, which postulates that a particular family of optimization problems have Gaussian extremizers (without time-division). We then show that this conjecture, if true, would imply the optimality of Gaussian inputs (with time-sharing) for the Han–Kobayashi achievable region for GZIC and further that the Han–Kobayashi achievable region matches the capacity region for GZIC. We also establish this conjecture in some regimes, which provides an outer bound to the slope of the capacity region at the Costa–Polyanskiy–Wu corner point.

1.3 Structure of this thesis

In Chapter 2 we consider the Han–Kobayashi achievable region with Gaussian inputs, that is, the region in Definition 1.3, and we show that any multi-letter extension, defined as in Definition 1.4, to the region, coincides with the single-letter region. This result implies that if the multi-letter Han–Kobayashi achievable region is attained by Gaussian inputs then the single-letter region with Gaussian inputs is the capacity region. This result first appeared in [NN19].

In Chapter 3 we consider two different settings. Firstly, for the Han–Kobayashi achievable region with Gaussian inputs for GZIC, we give a necessary and sufficient condition for a weighted sum-rate to be attained at the Costa–Sato corner point, hence giving the slope of the region at the corner point. This result first appeared in [CNN17]. Secondly, for the capacity region of GIC under the condition $a(1 + b^2P_2) + b(1 + a^2P_1) \leq 1$, we show that a collection of weighted sum-rates is attained at the maximum sum-rate point, which is also the treating-interference-as-noise point, implying a slope discontinuity at this point. This computation is new in this thesis as a slight generalization of the result in [Bei+16].

In Chapter 4 we propose a conjecture concerning Gaussian extremality of a functional. The chapter consists of two parts. In the first part we establish that the conjecture implies optimality of Han–Kobayashi achievable region for GZIC. This result first appeared in [Cos+20]. In the second part we show an information inequality that establishes the conjecture in some regimes. This inequality also implies that a collection of weighted sum-rates of the capacity region of GZIC is attained at the Costa–Polyanskiy–Wu corner point, yielding an improved outer bound for the slope at the corner point. This result first appeared in [GNN21].

Chapter 2

Multi-letter extension of Han–Kobayashi achievable region for GIC

The main result of this chapter is Theorem 2.1, which says that the multi-letter extension of the Han–Kobayashi achievable region with correlated vector Gaussian inputs matches the single-letter region with scalar Gaussian inputs for the GIC. The main ingredient of the proof is an inequality of Fiedler [Fie71] that enables one to bound the determinant of a sum of Hermitian matrices in terms of their eigenvalues and which generalizes the classical rearrangement inequality. Application of Fiedler's inequality yields upper bounds for each of the inequalities characterizing the multi-letter region by suitable diagonal covariance matrices, which in turn gives the single-letter region.

There have been attempts to study the local optimality of Gaussian distributions for the Han–Kobayashi achievable region with perturbations using Hermite polynomials [AZ12] as well as using temporally correlated coding schemes. While the former approach yielded interesting insights, so far the approach has not exhibited any rate pair that lays outside the Han–Kobayashi achievable region with Gaussian inputs. There have been some instances in network information theory where multi-letter Gaussian schemes have been shown to match the single-letter scheme, such as [CV93; Bei+16; CNN17]. The result of this chapter is a natural extension of such results and deals with the Han–Kobayashi achievable region in its entirety.

The result of this chapter first appeared in [NN19]. In a previous paper [CNN17] we have shown the result for the special case of Z-interference.

Theorem 2.1. $\mathcal{R}_n^{\textit{HK-GS}} = \mathcal{R}_1^{\textit{HK-GS}}$ *for any* $n \geq 1$.

2.1 Proof of reduction of multi-letter region to single-letter

Let $\lambda_i(A)$ denote the i-th ($i \geq 1$) smallest eigenvalue (counting multiplicity) of a Hermitian matrix A. We first state some standard technical results that the proof uses.

Theorem 2.2 (Fiedler [Fie71]). *Let A, B be $n \times n$ Hermitian matrices with $\lambda_1(A) + \lambda_1(B) \geq 0$. Then*

$$\prod_{i=1}^{n} (\lambda_i(A) + \lambda_i(B)) \leq |A + B| \leq \prod_{i=1}^{n} (\lambda_i(A) + \lambda_{n+1-i}(B)) .$$

Lemma 2.1. *Let A, B be $n \times n$ Hermitian matrices with $B \succeq 0$. Then $\lambda_i(A+B) \geq \lambda_i(A)$ for all $1 \leq i \leq n$.*

Proof. We have

$$\begin{aligned}
\lambda_i(A+B) &\overset{(a)}{=} \min_{\substack{V \subseteq \mathbb{C}^n \\ \dim V = i}} \max_{\substack{x \in V \\ \|x\|=1}} x^*(A+B)x \\
&\overset{(b)}{\geq} \min_{\substack{V \subseteq \mathbb{C}^n \\ \dim V = i}} \max_{\substack{x \in V \\ \|x\|=1}} x^* A x \\
&\overset{(a)}{=} \lambda_i(A),
\end{aligned}$$

where (a) is an application of the Courant-Fischer-Weyl min-max principle and (b) follows from $B \succeq 0$. □

Proof of Theorem 2.1. From its definition we can write $\mathcal{R}_n^{\text{HK-GS}}$ more explicitly as the set of rate pairs (R_1, R_2) satisfying

$$R_1 \leq \mathrm{E}_Q \left[\frac{1}{2n} \log \frac{\left| I + K^Q_{\mathbf{U}_1} + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{V}_2} \right|}{\left| I + b^2 K^Q_{\mathbf{V}_2} \right|} \right], \tag{2.1a}$$

$$R_2 \leq \mathrm{E}_Q \left[\frac{1}{2n} \log \frac{\left| I + K^Q_{\mathbf{U}_2} + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{V}_1} \right|}{\left| I + a^2 K^Q_{\mathbf{V}_1} \right|} \right], \tag{2.1b}$$

$$\begin{aligned}
R_1 + R_2 \leq \mathrm{E}_Q \Bigg[&\frac{1}{2n} \log \frac{\left| I + K^Q_{\mathbf{U}_1} + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{U}_2} + b^2 K^Q_{\mathbf{V}_2} \right|}{\left| I + b^2 K^Q_{\mathbf{V}_2} \right|} \\
&+ \frac{1}{2n} \log \frac{\left| I + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{V}_1} \right|}{\left| I + a^2 K^Q_{\mathbf{V}_1} \right|} \Bigg],
\end{aligned} \tag{2.1c}$$

$$R_1 + R_2 \leq \mathrm{E}_Q \Bigg[\frac{1}{2n} \log \frac{\left| I + K^Q_{\mathbf{U}_2} + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{U}_1} + a^2 K^Q_{\mathbf{V}_1} \right|}{\left| I + a^2 K^Q_{\mathbf{V}_1} \right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{V}_2}\right|}{\left|I + b^2 K^Q_{\mathbf{V}_2}\right|}\Bigg], \tag{2.1d}$$

$$R_1 + R_2 \le \mathrm{E}_Q \Bigg[\frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{U}_2} + b^2 K^Q_{\mathbf{V}_2}\right|}{\left|I + b^2 K^Q_{\mathbf{V}_2}\right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{U}_1} + a^2 K^Q_{\mathbf{V}_1}\right|}{\left|I + a^2 K^Q_{\mathbf{V}_1}\right|}\Bigg], \tag{2.1e}$$

$$2R_1 + R_2 \le \mathrm{E}_Q \Bigg[\frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{U}_1} + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{U}_2} + b^2 K^Q_{\mathbf{V}_2}\right|}{\left|I + b^2 K^Q_{\mathbf{V}_2}\right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{V}_2}\right|}{\left|I + b^2 K^Q_{\mathbf{V}_2}\right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{U}_1} + a^2 K^Q_{\mathbf{V}_1}\right|}{\left|I + a^2 K^Q_{\mathbf{V}_1}\right|}\Bigg], \tag{2.1f}$$

$$R_1 + 2R_2 \le \mathrm{E}_Q \Bigg[\frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{U}_2} + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{U}_1} + a^2 K^Q_{\mathbf{V}_1}\right|}{\left|I + a^2 K^Q_{\mathbf{V}_1}\right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{V}_1}\right|}{\left|I + a^2 K^Q_{\mathbf{V}_1}\right|}$$

$$+ \frac{1}{2n} \log \frac{\left|I + K^Q_{\mathbf{V}_1} + b^2 K^Q_{\mathbf{U}_2} + b^2 K^Q_{\mathbf{V}_2}\right|}{\left|I + b^2 K^Q_{\mathbf{V}_2}\right|}\Bigg], \tag{2.1g}$$

for some $p(q)$ and $n \times n$ positive semidefinite matrices $K^q_{\mathbf{U}_1}, K^q_{\mathbf{V}_1}, K^q_{\mathbf{U}_2}, K^q_{\mathbf{V}_1}$ (defined for every choice of $Q = q$) such that the power constraints

$$\mathrm{E}_Q \left[\mathrm{tr}\left(K^Q_{\mathbf{U}_1} + K^Q_{\mathbf{V}_1}\right)\right] \le nP_1,$$
$$\mathrm{E}_Q \left[\mathrm{tr}\left(K^Q_{\mathbf{U}_2} + K^Q_{\mathbf{V}_2}\right)\right] \le nP_2$$

are satisfied. By a standard application of cardinality-bounding techniques it suffices to consider $|\mathcal{Q}| \le 9$.

For any collection of $n \times n$ positive semidefinite matrices $K^q_{\mathbf{U}_1}, K^q_{\mathbf{V}_1}, K^q_{\mathbf{U}_2}, K^q_{\mathbf{V}_2}$ define

$$\hat{K}^q_{\mathbf{U}_1} := \mathrm{diag}\left(\{\lambda_i(K^q_{\mathbf{U}_1} + K^q_{\mathbf{V}_1}) - \lambda_i(K^q_{\mathbf{V}_1})\}\right),$$
$$\hat{K}^q_{\mathbf{V}_1} := \mathrm{diag}\left(\{\lambda_i(K^q_{\mathbf{V}_1})\}\right),$$
$$\hat{K}^q_{\mathbf{U}_2} := \mathrm{diag}\left(\{\lambda_{n+1-i}(K^q_{\mathbf{U}_2} + K^q_{\mathbf{V}_2}) - \lambda_{n+1-i}(K^q_{\mathbf{V}_2})\}\right),$$

$$\hat{K}^q_{\mathbf{V}_2} := \operatorname{diag}\left(\{\lambda_{n+1-i}(K^q_{\mathbf{V}_2})\}\right),$$

where $\operatorname{diag}(\{a_i\})$ indicates a diagonal matrix with diagonal entries $a_1, \ldots, a_n$. The positive semidefiniteness of $\hat{K}_{\mathbf{U}_1}, \hat{K}_{\mathbf{U}_2}$ follows from Lemma 2.1, and it is obvious that $\hat{K}^q_{\mathbf{V}_1}, \hat{K}^q_{\mathbf{V}_2}$ are positive semidefinite. This choice of the matrices $\hat{K}^q_{\mathbf{U}_1}, \hat{K}^q_{\mathbf{V}_1}, \hat{K}^q_{\mathbf{U}_2}, \hat{K}^q_{\mathbf{V}_2}$ has the following properties:

(i) The replacement is trace preserving, i.e.,

$$\begin{aligned}
\operatorname{tr}(K^q_{\mathbf{U}_1} + K^q_{\mathbf{V}_1}) &= \operatorname{tr}(\hat{K}^q_{\mathbf{U}_1} + \hat{K}^q_{\mathbf{V}_1}),\\
\operatorname{tr}(K^q_{\mathbf{U}_2} + K^q_{\mathbf{V}_2}) &= \operatorname{tr}(\hat{K}^q_{\mathbf{U}_2} + \hat{K}^q_{\mathbf{V}_2}).
\end{aligned}$$

(ii) The replacement preserves all the denominators of the terms in (2.1), i.e.,

$$\begin{aligned}
\left|I + a^2 K^q_{\mathbf{V}_1}\right| &= \left|I + a^2 \hat{K}^q_{\mathbf{V}_1}\right|,\\
\left|I + b^2 K^q_{\mathbf{V}_2}\right| &= \left|I + b^2 \hat{K}^q_{\mathbf{V}_2}\right|.
\end{aligned}$$

(iii) The replacement cannot decrease the numerators of the terms in (2.1): For any $c_1, c_2 \geq 0$, $(A_1, \hat{A}_1) = (K^q_{\mathbf{V}_1}, \hat{K}^q_{\mathbf{V}_1})$ or $(K^q_{\mathbf{U}_1} + K^q_{\mathbf{V}_1}, \hat{K}^q_{\mathbf{U}_1} + \hat{K}^q_{\mathbf{V}_1})$, and $(A_2, \hat{A}_2) = (K^q_{\mathbf{V}_2}, \hat{K}^q_{\mathbf{V}_2})$ or $(K^q_{\mathbf{U}_2} + K^q_{\mathbf{V}_2}, \hat{K}^q_{\mathbf{U}_2} + \hat{K}^q_{\mathbf{V}_2})$, we have

$$\begin{aligned}
|I + c_1 A_1 + c_2 A_2| &\leq \prod_{i=1}^{n} \left(1 + c_1 \lambda_i(A_1) + c_2 \lambda_{n+1-i}(A_2)\right)\\
&= \left|I + c_1 \hat{A}_1 + c_2 \hat{A}_2\right|,
\end{aligned}$$

where the inequality follows from Theorem 2.2.

Therefore replacing $(K^q_{\mathbf{U}_1}, K^q_{\mathbf{V}_1}, K^q_{\mathbf{U}_2}, K^q_{\mathbf{V}_2})$ by $(\hat{K}^q_{\mathbf{U}_1}, \hat{K}^q_{\mathbf{V}_1}, \hat{K}^q_{\mathbf{U}_2}, \hat{K}^q_{\mathbf{V}_2})$ cannot decrease any of the right-hand sides of (2.1). This shows that $\mathcal{R}_n^{\text{HK-GS}}$ can be attained by further assuming that the covariance matrices $K^q_{\mathbf{U}_1}, K^q_{\mathbf{V}_1}, K^q_{\mathbf{U}_2}, K^q_{\mathbf{V}_2}$ are diagonal for every q.

When the matrices $K^q_{\mathbf{U}_1}, K^q_{\mathbf{V}_1}, K^q_{\mathbf{U}_2}, K^q_{\mathbf{V}_2}$ are diagonal with diagonal entries $K^q_{\mathbf{U}_1}(i), K^q_{\mathbf{V}_1}(i), K^q_{\mathbf{U}_2}(i), K^q_{\mathbf{V}_2}(i)$ $(i = 1, \ldots, n)$, respectively, observe that, for instance, we can express

$$\begin{aligned}
&\mathrm{E}_Q\left[\frac{1}{2n}\log\left|I + K^Q_{\mathbf{U}_2} + K^Q_{\mathbf{V}_2} + a^2 K^Q_{\mathbf{U}_1} + a^2 K^Q_{\mathbf{V}_1}\right|\right]\\
&= \sum_q \mathrm{P}(Q = q)\left(\frac{1}{2n}\sum_{i=1}^{n}\log\left(1 + K^q_{\mathbf{U}_2}(i) + K^q_{\mathbf{V}_2}(i) + a^2 K^q_{\mathbf{U}_1}(i) + a^2 K^q_{\mathbf{V}_1}(i)\right)\right)\\
&= \sum_{q,i} \mathrm{P}(\tilde{Q} = (q,i))\left(\frac{1}{2}\log\left(1 + K^{(q,i)}_{\mathbf{U}_2} + K^{(q,i)}_{\mathbf{V}_2} + a^2 K^{(q,i)}_{\mathbf{U}_1} + a^2 K^{(q,i)}_{\mathbf{V}_1}\right)\right)
\end{aligned}$$

$$= \mathrm{E}_{\tilde{Q}} \left[\frac{1}{2} \log \left(1 + K_{\mathbf{U}_2}^{\tilde{Q}} + K_{\mathbf{V}_2}^{\tilde{Q}} + a^2 K_{\mathbf{U}_1}^{\tilde{Q}} + a^2 K_{\mathbf{V}_1}^{\tilde{Q}} \right) \right],$$

where we have defined a new time-sharing variable $\tilde{Q}$ taking values in $\mathcal{Q} \times \{1, \ldots, n\}$ by setting $\mathrm{P}(\tilde{Q} = (q,i)) := \frac{1}{n} \mathrm{P}(Q = q)$ as well as scalar variables $K_{\mathbf{U}_1}^{(q,i)} := K_{\mathbf{U}_1}^{q}(i)$ and similarly for the others. Note that the last expression is an expectation over scalar variables and corresponds to the expression in $\mathcal{R}_1^{\text{HK-GS}}$. All the other terms in (2.1) as well as the trace constraints can also be expressed similarly. Now we can conclude the inclusion $\mathcal{R}_n^{\text{HK-GS}} \subseteq \mathcal{R}_1^{\text{HK-GS}}$. Together with the trivial inclusion $\mathcal{R}_1^{\text{HK-GS}} \subseteq \mathcal{R}_n^{\text{HK-GS}}$ this establishes Theorem 2.1. $\square$

Chapter 3

Optimality of weighted sum-rates

This chapter consists of two parts. In the first part we compute the slope of Han–Kobayashi achievable region with Gaussian inputs for GZIC at the Costa–Sato corner point, which is also the maximum sum-rate point. This result first appeared in [CNN17]. The computation uses similar techniques as in an earlier paper [CN16] in which its authors compute the slope of the same region at the Costa–Polyanskiy–Wu corner point. In the second part we focus on the capacity region of GIC under the condition $a(1+b^2P_2)+b(1+a^2P_1) \le 1$, which is a subclass of the weak interference regime, and we establish the discontinuity of slope at the maximum sum-rate point, which is also the treating-interference-as-noise point:

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{P_1}{1+b^2P_2}\right), \frac{1}{2}\log\left(1+\frac{P_2}{1+a^2P_1}\right)\right).$$

This result slightly generalizes the computation in [Bei+16], where the authors show the same result under symmetric interference condition.

3.1 Slope at maximum sum-rate point for Han–Kobayashi achievable region with Gaussian inputs of GZIC

For the GZIC, consider the multi-letter Han–Kobayashi achievable region with Gaussian inputs, $\mathcal{R}_n^{\text{HK-GS}}$, as defined in Definition 1.4. In view of Theorem 2.1, this region is the same for all $n \ge 1$ and so it is the same as the single-letter region $\mathcal{R}^{\text{HK-GS}}$. Obviously $\mathcal{R}^{\text{HK-GS}}$ contains the Costa–Sato corner point of the capacity region. In this section we establish Theorem 3.1, which concerns the slope of $\mathcal{R}^{\text{HK-GS}}$ around the Costa–Sato corner point.

Theorem 3.1. *For a GZIC, let $\lambda_{critical}$ be the largest value of $\lambda \geq 1$ such that*

$$\sup_{(R_1,R_2)\in\mathcal{R}^{HK\text{-}GS}} (R_1 + \lambda R_2) = \frac{1}{2}\log(1+P_1) + \frac{\lambda}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right). \tag{3.1}$$

Then

$$\lambda_{critical} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \lambda^*\right\},$$

where λ^* is the unique positive solution of $\psi(\lambda^*) = 0$, where

$$\psi(\lambda) := \lambda\left(\log\left(1+\frac{P_2}{1+a^2P_1}\right) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right) + \log\left(1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\lambda\right).$$

The region $\mathcal{R}^{\text{HK-GS}}$ admits a reduction as given in Proposition 3.1 in a similar manner to Proposition 1.2. To compute $\sup_{(R_1,R_2)\in\mathcal{R}^{\text{HK-GS}}}(R_1 + \lambda R_2)$ for $\lambda \geq 1$ we notice that $\mathcal{R}^{\text{HK-GS}}$ is a union of pentagons and (3.2b) and (3.2c) are the only two tight constraints. We then parametrize $U_1|_{Q=q} \sim \mathcal{N}(0, (1-\alpha_q)K_{1q})$, $V_1|_{Q=q} \sim \mathcal{N}(0, \alpha_q K_{1q})$, $X_2|_{Q=q} \sim \mathcal{N}(0, K_{2q})$, where $K_{1q}, K_{2q} \geq 0$ and $0 \leq \alpha_q \leq 1$ for each choice of q. With this parametrization, the power constraints (3.3) read $\mathrm{E}_Q[K_{1Q}] \leq P_1$ and $\mathrm{E}_Q[K_{2Q}] \leq P_2$. Then

$$\begin{aligned}
&\sup_{(R_1,R_2)\in\mathcal{R}^{\text{HK-GS}}} (R_1 + \lambda R_2) \\
&= \sup_{p(q)p(u_1|q)p(v_1|q)p(x_2|q)} (I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q) + (\lambda-1)I(X_2; Y_2|U_1, Q)) \\
&= \sup_{p(q)p(u_1|q)p(v_1|q)p(x_2|q)} (h(X_2 + aU_1 + aV_1 + Z_2|Q) - h(aV_1 + Z_2|Q) \\
&\quad + h(V_1 + Z_1|Q) - h(Z_1) + (\lambda-1)(h(X_2 + aV_1 + Z_2|Q) - h(aV_1 + Z_2|Q))) \\
&= \sup_{p(q),K_{1q},K_{2q},\alpha_q} \mathrm{E}_Q\left[\frac{1}{2}\log(1 + a^2K_{1Q} + K_{2Q})\right. \\
&\quad \left. + \frac{\lambda}{2}\log\frac{1 + a^2\alpha_Q K_{1Q} + K_{2Q}}{1 + a^2\alpha_Q K_{1Q}} + \frac{1}{2}\log\frac{1+\alpha_Q K_{1Q}}{1 + a^2\alpha_Q K_{1Q} + K_{2Q}}\right] \\
&= \sup_{p(q),K_{1q},K_{2q}} \mathrm{E}_Q[f_\lambda(K_{1Q}, K_{2Q})],
\end{aligned}$$

where f_λ is defined as in Corollary 3.1. The supremum in the last line is subjected to the power constraints and hence is evaluated to the value of upper concave envelope of f_λ at (P_1, P_2). This yields Corollary 3.1.

Proposition 3.1. *The region $\mathcal{R}^{HK\text{-}GS}$ for GZIC is equal to the set of all rate pairs (R_1, R_2) such that*

$$R_1 \leq I(X_1; Y_1|Q), \tag{3.2a}$$

$$R_2 \le I(X_2; Y_2 | U_1, Q), \tag{3.2b}$$

$$R_1 + R_2 \le I(U_1, X_2; Y_2 | Q) + I(X_1; Y_1 | U_1, Q), \tag{3.2c}$$

where $X_1 := U_1 + V_1$, for some $p(q)p(u_1|q)p(v_1|q)p(x_2|q)$ such that the conditional distributions $p(u_1|q), p(v_1|q), p(x_2|q)$ are zero-mean scalar Gaussian distributions for each q, and that the power constraints

$$\mathrm{E}_Q\left[\mathrm{Cov}(U_1|Q) + \mathrm{Cov}(V_1|Q)\right] \le P_1, \tag{3.3a}$$

$$\mathrm{E}_Q\left[\mathrm{Cov}(X_2|Q)\right] \le P_2 \tag{3.3b}$$

are satisfied.

Corollary 3.1. *For a GZIC, the value of $\sup_{(R_1,R_2)\in\mathcal{R}^{HK\text{-}GS}}(R_1 + \lambda R_2)$ for $\lambda \ge 1$ is equal to the upper concave envelope of the functional f_λ evaluated at (P_1, P_2), where*

$$f_\lambda(Q_1, Q_2) := \frac{1}{2}\log(1 + a^2 Q_1 + Q_2) + \max_{0\le\alpha\le 1}\left(\frac{\lambda}{2}\log\frac{1 + a^2\alpha Q_1 + Q_2}{1 + a^2\alpha Q_1} + \frac{1}{2}\log\frac{1 + \alpha Q_1}{1 + a^2\alpha Q_1 + Q_2}\right)$$

for $Q_1, Q_2 \ge 0$.

Consider the functional f_λ defined in Corollary 3.1. By taking derivative with respect to α, we get the first-order condition for the optimal value α^* for α:

$$\lambda = \frac{1 - a^2 + Q_2}{a^2 Q_2}\left(a^2 + \frac{1 - a^2}{1 + \alpha^* Q_1}\right).$$

We then define the regions

$$\mathcal{R}_1 := \left\{(Q_1, Q_2) \in \mathbb{R}^2_{\ge 0} : \lambda \ge \frac{1 - a^2 + Q_2}{a^2 Q_2}\right\},$$
$$\mathcal{R}_2 := \left\{(Q_1, Q_2) \in \mathbb{R}^2_{\ge 0} : \lambda \le \frac{(1 - a^2 + Q_2)(1 + a^2 Q_1)}{a^2 Q_2 (1 + Q_1)}\right\},$$
$$\mathcal{R}_3 := \left\{(Q_1, Q_2) \in \mathbb{R}^2_{\ge 0} : \frac{(1 - a^2 + Q_2)(1 + a^2 Q_1)}{a^2 Q_2 (1 + Q_1)} < \lambda < \frac{1 - a^2 + Q_2}{a^2 Q_2}\right\},$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ correspond to the cases $\alpha^* = 0$, $\alpha^* = 1$ and $0 < \alpha^* < 1$, respectively. This gives an explicit expression for f_λ:

$$f_\lambda(Q_1, Q_2) = \begin{cases} \frac{1}{2}\log(1 + a^2 Q_1 + Q_2) + \frac{\lambda - 1}{2}\log(1 + Q_2) & \text{if } (Q_1, Q_2) \in \mathcal{R}_1, \\ \frac{\lambda}{2}\log(1 + \frac{Q_2}{1 + a^2 Q_1}) + \frac{1}{2}\log(1 + Q_1) & \text{if } (Q_1, Q_2) \in \mathcal{R}_2, \\ \frac{1}{2}\log\frac{1 + a^2 Q_1 + Q_2}{a^2 Q_2} + \frac{\lambda - 1}{2}\log(\lambda - 1) - \frac{\lambda}{2}\log\lambda & \\ \qquad + \frac{\lambda}{2}\log\frac{1 - a^2 + Q_2}{1 - a^2} + \frac{1}{2}\log(1 - a^2) & \text{if } (Q_1, Q_2) \in \mathcal{R}_3. \end{cases}$$

Now we compute the gradient and Hessian of f_λ. In $\mathcal{R}_1$,

$$\partial_{Q_1} f_\lambda = \frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2},$$

$$\partial_{Q_2} f_\lambda = \frac{1}{2}\frac{1}{1+a^2Q_1+Q_2} + \frac{\lambda-1}{2}\frac{1}{1+Q_2},$$

$$\mathcal{H} f_\lambda = \begin{pmatrix} \frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-1}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} - \frac{\lambda-1}{2}\frac{1}{(1+Q_2)^2} \end{pmatrix}.$$

In $\mathcal{R}_2$,

$$\partial_{Q_1} f_\lambda = \frac{1}{2}\left(\frac{a^2\lambda}{1+a^2Q_1+Q_2} - \frac{a^2\lambda}{1+a^2Q_1} + \frac{1}{1+Q_1}\right),$$

$$\partial_{Q_2} f_\lambda = \frac{\lambda}{2}\frac{1}{1+a^2Q_1+Q_2},$$

$$\mathcal{H} f_\lambda = \begin{pmatrix} \frac{1}{2}\left(\frac{-a^4\lambda}{(1+a^2Q_1+Q_2)^2} + \frac{a^4\lambda}{(1+a^2Q_1)^2} - \frac{1}{(1+Q_1)^2}\right) & \frac{-a^2\lambda}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2\lambda}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-\lambda}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \end{pmatrix}.$$

In $\mathcal{R}_3$,

$$\partial_{Q_1} f_\lambda = \frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2},$$

$$\partial_{Q_2} f_\lambda = \frac{1}{2}\left(\frac{1}{1+a^2Q_1+Q_2} - \frac{1}{Q_2} + \frac{\lambda}{1-a^2+Q_2}\right),$$

$$\mathcal{H} f_\lambda = \begin{pmatrix} \frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{1}{2}\left(\frac{-1}{(1+a^2Q_1+Q_2)^2} + \frac{1}{Q_2^2} - \frac{\lambda}{(1-a^2+Q_2)^2}\right) \end{pmatrix}.$$

By checking the values and gradients of f_λ at the boundaries, one can see that f_λ is continuously differentiable on $\mathbb{R}^2_{>0}$.

For λ slightly larger than $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$, we have $(P_1,P_2) \in \mathcal{R}_3$. The function

$$\lambda \mapsto f_\lambda(P_1,P_2) - \left(\frac{\lambda}{2}\log(1+\frac{P_2}{1+a^2P_1}) + \frac{1}{2}\log(1+P_1)\right)$$

is equal to 0 and has derivative equal to 0 at $\lambda = \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$, and has second derivative larger than 0 for λ slightly larger than $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$. So we have

$$f_\lambda(P_1,P_2) > \frac{\lambda}{2}\log(1+\frac{P_2}{1+a^2P_1}) + \frac{1}{2}\log(1+P_1)$$

for λ slightly larger than $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$. Thus any λ satisfying (3.1) must also satisfy $\lambda \le \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$, or equivalently $(P_1,P_2) \in \mathcal{R}_2$.

Using Corollary 3.1 and Lemma 3.1, for any $\lambda \ge 1$, (3.1) is equivalent to that $(P_1,P_2) \in \mathcal{R}_2$ and $g_\lambda(Q_1,Q_2)$ attains global maximum at (P_1,P_2), where g_λ is defined by

$$g_\lambda(Q_1,Q_2) := f_\lambda(Q_1,Q_2) - \frac{1}{2}\left(\frac{a^2\lambda}{1+a^2P_1+P_2} - \frac{a^2\lambda}{1+a^2P_1} + \frac{1}{1+P_1}\right)Q_1$$

$$-\frac{1}{2}\left(\frac{\lambda}{1+a^2P_1+P_2}\right)Q_2.$$

The proof of Theorem 3.1 is completed by analyzing the local behavior of g_λ and isolating the local maxima.

Lemma 3.1. *Let f be a real-valued function defined on some convex subset of $\mathbb{R}^n$ $(n \geq 1)$. Suppose f is differentiable at $x \in \mathbb{R}^n$. Then $\mathcal{C}f(x) = f(x)$ if and only if $f(\cdot) - \langle \nabla f(x), \cdot \rangle$ attains global maximum at x. Here $\mathcal{C}f$ and ∇f denote the upper concave envelope and gradient of f, respectively.*

Proof. It suffices to show that $\mathcal{C}f(x) \leq f(x)$ if and only if for all h we have $f(x) \geq f(x+h) - \langle \nabla f(x), h\rangle$. The "if" part is immediate, by taking upper concave envelope with respect to h and then putting $h = 0$.

For the "only if" part, suppose on the contrary that there is $\epsilon > 0$ and $h \neq 0$ such that

$$f(x) + \epsilon \leq f(x+h) - \langle \nabla f(x), h \rangle .$$

By differentiability of f at x, for $\|\zeta\|$ small enough we have

$$|f(x+\zeta) - f(x) - \langle \nabla f(x), \zeta\rangle| \leq \frac{\epsilon}{2\|h\|}\|\zeta\|.$$

Now, for any $\delta \in (0,1)$ we have

$$\begin{aligned}
f(x) &\geq \mathcal{C}f(x) \\
&\geq \delta \cdot \mathcal{C}f(x+h) + (1-\delta)\cdot \mathcal{C}f(x - \frac{\delta}{1-\delta}h) \\
&\geq \delta f(x+h) + (1-\delta) f(x - \frac{\delta}{1-\delta}h) \\
&\geq \delta\epsilon + \delta f(x) + \langle \nabla f(x), \delta h\rangle + (1-\delta) f(x - \frac{\delta}{1-\delta}h).
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
f(x) &\geq \frac{\delta}{1-\delta}\epsilon + f(x - \frac{\delta}{1-\delta}h) - \left\langle \nabla f(x), -\frac{\delta}{1-\delta}h \right\rangle \\
&\geq \frac{\delta}{1-\delta}\epsilon + f(x) - \frac{\epsilon}{2\|h\|}\left\|-\frac{\delta}{1-\delta}h\right\| \\
&= f(x) + \frac{\epsilon}{2}\frac{\delta}{1-\delta},
\end{aligned}$$

for δ small enough. This gives a contradiction. $\square$

Interior analysis

Lemma 3.2. *Let $\lambda_{critical}$ be defined as in Theorem 3.1. Then*

$$\lambda_{critical} \leq \min\left\{ \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 \right\}.$$

Proof. This is the condition that says $(P_1, P_2) \in \mathcal{R}_2$ and it is a local maximum of g_λ. $\lambda_{\text{critical}} \le \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$ since $(P_1, P_2) \in \mathcal{R}_2$, and the second condition $\lambda_{\text{critical}} \le \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$ follows from $|\mathcal{H}g_\lambda(P_1, P_2)| \ge 0$. $\square$

Lemma 3.3. *There is no local maximum of g_λ in the interior of $\mathcal{R}_1$ for any $\lambda \le \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. Since g_λ is concave in $\mathcal{R}_1$, there is at most one local maximum in the interior of $\mathcal{R}_1$. The first-order condition yields

$$\frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2} = \frac{1}{2}\left(\frac{a^2\lambda}{1+a^2P_1+P_2} - \frac{a^2\lambda}{1+a^2P_1} + \frac{1}{1+P_1}\right),$$

$$\frac{1}{2}\frac{1}{1+a^2Q_1+Q_2} + \frac{\lambda-1}{2}\frac{1}{1+Q_2} = \frac{1}{2}\left(\frac{\lambda}{1+a^2P_1+P_2}\right).$$

Solving for Q_2 gives

$$Q_2 = \frac{(1+a^2P_1)(1+P_1)}{1+P_1+\frac{1-\frac{1}{a^2}}{\lambda-1}}.$$

But in $\mathcal{R}_1$ we have $\lambda \ge \frac{1-a^2+Q_2}{a^2Q_2}$. Substituting for Q_2 yields

$$\lambda \ge \frac{1+a^2P_1}{a^4(1+P_1)}.$$

But we also have $\lambda \le \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, implying $a^2 \ge 1$. This gives a contradiction. $\square$

Lemma 3.4. *There are at most 2 local maxima of g_λ in the interior of $\mathcal{R}_2$. The value of g_λ at both points is bounded from above by $g_\lambda(P_1, P_2)$, when $\lambda \le \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. The first-order condition for local maximum yields

$$\frac{a^2\lambda}{1+a^2Q_1+Q_2} - \frac{a^2\lambda}{1+a^2Q_1} + \frac{1}{1+Q_1} = \frac{a^2\lambda}{1+a^2P_1+P_2} - \frac{a^2\lambda}{1+a^2P_1} + \frac{1}{1+P_1},$$

$$\frac{\lambda}{1+a^2Q_1+Q_2} = \frac{\lambda}{1+a^2P_1+P_2},$$

whose solutions are given by

$$Q_1 = P_1 \text{ or } \frac{\frac{1}{a^2}-1}{\frac{\lambda}{k}-1} - 1,$$

$$Q_2 = P_2 + a^2(P_1 - Q_1),$$

where $k := \frac{1+a^2P_1}{a^2(1+P_1)} \ge 1$. If $k \ge \lambda$, there is only one solution at (P_1, P_2), so we can assume $k < \lambda$.

Suppose (Q_1, Q_2) is any solution to the above, then

$$\begin{aligned}
g_\lambda(Q_1, Q_2) &= f_\lambda(Q_1, Q_2) - \frac{\lambda}{2}\frac{a^2Q_1 + Q_2}{1 + a^2Q_1 + Q_2} + \frac{\lambda}{2}\frac{a^2Q_1}{1 + a^2Q_1} - \frac{1}{2}\frac{Q_1}{1 + Q_1} \\
&= f_\lambda(Q_1, Q_2) + \frac{\lambda}{2}\frac{1}{1 + a^2Q_1 + Q_2} - \frac{\lambda}{2}\frac{1}{1 + a^2Q_1} + \frac{1}{2}\frac{1}{1 + Q_1} - \frac{1}{2} \\
&= \frac{\lambda}{2}\varphi\left(1 + a^2Q_1 + Q_2\right) - \frac{\lambda}{2}\varphi\left(1 + a^2Q_1\right) + \frac{1}{2}\varphi\left(1 + Q_1\right) - \frac{1}{2} \\
&= \frac{\lambda}{2}\varphi\left(1 + a^2P_1 + P_2\right) - \frac{\lambda}{2}\varphi\left(1 + a^2Q_1\right) + \frac{1}{2}\varphi\left(1 + Q_1\right) - \frac{1}{2},
\end{aligned}$$

where $\varphi(x) := \log x + \frac{1}{x}$.

Now let (Q_1, Q_2) be the solution other than (P_1, P_2). Then,

$$\begin{aligned}
&g_\lambda(P_1, P_2) - g_\lambda(Q_1, Q_2) \\
&= \frac{\lambda}{2}\left(\varphi\left(1 + a^2Q_1\right) - \varphi\left(1 + a^2P_1\right)\right) - \frac{1}{2}\left(\varphi\left(1 + Q_1\right) - \varphi\left(1 + P_1\right)\right) \\
&= \frac{\lambda}{2}\left(\varphi\left((1 - a^2)\frac{\lambda}{\lambda - k}\right) - \varphi\left((1 - a^2)\frac{k}{k - 1}\right)\right) \\
&\quad - \frac{1}{2}\left(\varphi\left(\frac{1 - a^2}{a^2}\frac{k}{\lambda - k}\right) - \varphi\left(\frac{1 - a^2}{a^2}\frac{1}{k - 1}\right)\right).
\end{aligned}$$

Differentiating with respect to λ and simplifying gives

$$\partial_\lambda\left(g_\lambda(P_1, P_2) - g_\lambda(Q_1, Q_2)\right) = \frac{1}{2}\left(\log\left(1 + \frac{k^2 - \lambda}{k(\lambda - k)}\right) - \frac{k^2 - \lambda}{k(\lambda - k)}\right).$$

This implies

$$\partial_\lambda\left(g_\lambda(P_1, P_2) - g_\lambda(Q_1, Q_2)\right) \leq 0,$$

since $\lambda \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 = k^2$ and $\log(1 + x) \leq x$ for all $x \geq 0$. So

$$g_\lambda(P_1, P_2) - g_\lambda(Q_1, Q_2) \geq g_{k^2}(P_1, P_2) - g_{k^2}(Q_1, Q_2) = 0$$

and hence $g_\lambda(P_1, P_2) \geq g_\lambda(Q_1, Q_2)$. $\square$

Lemma 3.5. *There is no local maximum of g_λ in the interior of $\mathcal{R}_3$ when $\lambda < \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. The first-order condition for local maximum yields

$$\begin{aligned}
\frac{a^2}{1 + a^2Q_1 + Q_2} &= \frac{a^2\lambda}{1 + a^2P_1 + P_2} - \frac{a^2\lambda}{1 + a^2P_1} + \frac{1}{1 + P_1}, \\
\frac{1}{1 + a^2Q_1 + Q_2} - \frac{1}{Q_2} + \frac{\lambda}{1 - a^2 + Q_2} &= \frac{\lambda}{1 + a^2P_1 + P_2}.
\end{aligned}$$

Substituting the first equation into the second gives

$$Q_2 = a^2(1 + P_1),$$

while the second-order condition $|\mathcal{H}f_\lambda(Q_1,Q_2)| \geq 0$ is equivalent to

$$\begin{aligned}\lambda &\geq \left(\frac{1-a^2+Q_2}{Q_2}\right)^2\\ &= \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2,\end{aligned}$$

which contradicts with that $\lambda < \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$. □

Thus the interior analysis shows that

$$\lambda \leq \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}$$

if and only if the value of $g_\lambda(Q_1,Q_2)$ at any interior local maximum is bounded from above by $g_\lambda(P_1,P_2)$ and $(P_1,P_2) \in \mathcal{R}_2$. The necessity follows from Lemma 3.3, 3.4, and 3.5 while sufficiency follows from Lemma 3.2.

Boundary analysis

The remaining cases are the boundaries $Q_1 = 0$ and $Q_2 = 0$. In this part, we first establish that $g_\lambda(P_1,P_2) \geq g_\lambda(Q_1,Q_2)$ for any (Q_1,Q_2) on the boundaries if and only if λ is smaller than or equal to the upper bound in Lemma 3.2 and λ^* in Theorem 3.1. Then in Lemma 3.9 we reduce the minimum of three terms to that of two of them, yielding $\lambda_{\text{critical}}$ in Theorem 3.1.

Lemma 3.6.

$$\frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1} \geq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2.$$

Proof. This is equivalent to

$$a^4\varphi\left(P_1\right) - \varphi\left(a^2P_1\right) \geq 0,$$

where $\varphi\left(x\right) := (1+x)^2\log(1+x) - (1+x)x$. Let

$$\psi(x) := a^4\varphi(x) - \varphi(a^2x).$$

Note that

$$\begin{aligned}\varphi'\left(x\right) &= 2(1+x)\log(1+x) - x,\\ \varphi''\left(x\right) &= 2\log(1+x) + 1,\end{aligned}$$

as well as

$$\psi'(x) = a^4\varphi'\left(x\right) - a^2\varphi'\left(a^2x\right),$$

$$\psi''(x) = a^4 \cdot 2\log\frac{1+x}{1+a^2x} \geq 0.$$

So $\psi(x)$ is convex when $x \geq 0$ and $\psi'(0) = 0$. This implies that ψ is convex and increasing on $x \geq 0$. Hence $\psi(P_1) \geq \psi(0) = 0$. $\square$

Lemma 3.7. *Suppose*

$$\lambda \leq \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}.$$

Then

$$g_\lambda(P_1, P_2) \geq \max_{Q_2 \geq 0} g_\lambda(0, Q_2).$$

Proof. We have

$$g_\lambda(0, Q_2) = \frac{\lambda}{2}\log(1+Q_2) - \frac{\lambda}{2}\frac{1}{1+a^2P_1+P_2}Q_2.$$

Note that $g_\lambda(0, Q_2)$ is concave in Q_2 and is maximized at $Q_2 = a^2P_1 + P_2$. Since $(P_1, P_2) \in \mathcal{R}_2$ we have

$$\begin{aligned}
&g_\lambda(P_1, P_2) - \max_{Q_2 \geq 0} g_\lambda(0, Q_2) \\
&= g_\lambda(P_1, P_2) - g_\lambda(0, a^2P_1 + P_2) \\
&= -\frac{\lambda}{2}\left(\log(1+a^2P_1) + \frac{1}{1+a^2P_1} - 1\right) + \frac{1}{2}\left(\log(1+P_1) + \frac{1}{1+P_1} - 1\right),
\end{aligned}$$

which is ≥ 0 if and only if $\lambda \leq \frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1}$, which is guaranteed by Lemma 3.6. $\square$

Lemma 3.8. *Suppose*

$$\lambda \leq \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}.$$

Then

$$g_\lambda(P_1, P_2) \geq \max_{Q_1 \geq 0} g_\lambda(Q_1, 0)$$

if and only if $\lambda \leq \lambda^*$, where λ^* is defined as in Theorem 3.1.

Proof. We have

$$g_\lambda(Q_1, 0) = \frac{1}{2}\log(1+Q_1) - \frac{1}{2}\left(\frac{a^2\lambda}{1+a^2P_1+P_2} - \frac{a^2\lambda}{1+a^2P_1} + \frac{1}{1+P_1}\right)Q_1.$$

Note that $g_\lambda(Q_1, 0)$ is concave in Q_1 and is maximized when

$$\frac{1}{1+Q_1} = \frac{a^2\lambda}{1+a^2P_1+P_2} - \frac{a^2\lambda}{1+a^2P_1} + \frac{1}{1+P_1}.$$

The right-hand side is between 0 and 1 since $\lambda \leq \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$ and hence such a maximizing $Q_1 \geq 0$ always exists. After some manipulations, we can express

$$\begin{aligned}
& g_\lambda(P_1, P_2) - \max_{Q_1 \geq 0} g_\lambda(Q_1, 0) \\
&= \frac{1}{2}\Bigg(\lambda\left(\log\left(1+\frac{P_2}{1+a^2P_1}\right) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right) \\
&\qquad + \log\left(1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\lambda\right)\Bigg) \\
&= \frac{1}{2}\psi(\lambda),
\end{aligned}$$

where the function ψ is defined as in Theorem 3.1.

The function ψ is concave, is equal to 0 at 0, and has non-negative derivative at 0. It follows that $\psi(\lambda^*) = 0$ has a unique positive solution λ^*, $\psi(\lambda) > 0$ for $0 < \lambda < \lambda^*$, and $\psi(\lambda) < 0$ for $\lambda > \lambda^*$. Hence $g_\lambda(P_1, P_2) \geq \max_{Q_1 \geq 0} g_\lambda(Q_1, 0)$ if and only if $\lambda \leq \lambda^*$. $\square$

Thus combining the interior and boundary analysis, we see that $\lambda_{\text{critical}}$ is the minimum of three quantities, two of which is given by Lemma 3.2 and one given by Lemma 3.8. Finally the proof of Theorem 3.1 is concluded by Lemma 3.9 which shows that one of the three quantities is redundant.

Lemma 3.9. *The following holds:*

$$\begin{aligned}
& \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \lambda^*\right\} \\
&= \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2, \lambda^*\right\},
\end{aligned}$$

where λ^* is defined as in Theorem 3.1.

Proof. It suffices to show that, if $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)} \geq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, or equivalently $P_2 \leq a^2(1+P_1)$, then $\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 \geq \lambda^*$. That is, $\psi\left(\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right) \leq 0$, where $\psi(\cdot)$ is defined in Theorem 3.1.

Let $\theta := \frac{P_2}{1+a^2P_1}$ and $k := \frac{1+a^2P_1}{a^2(1+P_1)}$. Then $\theta \leq \frac{1}{k}$ and $k \geq 1$. We want to show $\psi(k^2) \leq 0$, that is,

$$k^2\log(1+\theta) - k(k-1)\frac{\theta}{\theta+1} + \log(1 - \frac{k\theta}{1+\theta}) \leq 0,$$

or equivalently

$$k^2 \left(\log(1+\theta) + \frac{1}{1+\theta} - 1 \right) + \left(\frac{k\theta}{1+\theta} + \log(1 - \frac{k\theta}{1+\theta}) \right) \le 0. \tag{3.4}$$

The derivative of left-hand side of (3.4) with respect to θ is equal to

$$\begin{aligned}
& k^2 \frac{\theta}{(1+\theta)^2} + \frac{k}{(1+\theta)^2} \frac{k\theta}{k\theta - (1+\theta)} \\
&= \frac{k^2\theta}{(1+\theta)^2} \left(1 + \frac{1}{k\theta - (1+\theta)} \right) \\
&= \frac{k^2\theta^2}{(1+\theta)^2} \frac{1-k}{1+\theta-k\theta} \\
&\le 0.
\end{aligned}$$

Hence it suffices to establish (3.4) with $\theta = 0$. Note that the left-hand side of (3.4) is equal to 0 when $\theta = 0$. So we are done. □

This completes the proof of Theorem 3.1.

3.2 Slope discontinuity at maximum sum-rate point for a subset of GIC

The following Theorem 3.4 is an extension to Theorem 1 in [Bei+16]. In [Bei+16] the authors showed the special case where the GIC has symmetric interference ($a = b$ and $P_1 = P_2$) while a similar proof can also be applied to a subset of the weak interference regime. The proof uses a genie-based outer bound [Liu16], originally developed in [Bei+16], for discrete memoryless interference channels.

To complete the proof of Theorem 3.4 we will also need Corollary 3.2, which follows from Theorem 3.3 (Theorem 1 of [GN14], originally established in [LV07]) concerning Gaussian optimality of a functional.

Theorem 3.2 (Theorem 2.1.1 of [Liu16])**.** *Consider a discrete memoryless interference channel $p(y_1, y_2|x_1, x_2)$. Let T_1, T_2 be any random variables such that the joint distribution satisfies*

$$p(y_1, y_2, t_1, t_2|x_1, x_2) = p(t_1|x_1)p(t_2|x_2)p(y_1, y_2|t_1, t_2, x_1, x_2),$$

and that the marginal distribution $p(y_1, y_2|x_1, x_2)$ is consistent with the interference channel. Then

$$\begin{aligned}
\sup_{(R_1,R_2)\in\mathcal{C}} (R_1 + \lambda R_2) \le \sup_{p(x_1)p(x_2)} \big(& I(X_1; T_1, Y_1) + \lambda I(X_2; T_2, Y_2) \\
& + \mathcal{C}_{p(x_1)p(x_2)}[I(X_1; T_1|X_2, T_2) - \lambda I(X_1; Y_2|X_2, T_2)]
\end{aligned}$$

$$
\begin{aligned}
&- (I(X_1;T_1|X_2,T_2) - \lambda I(X_1;Y_2|X_2,T_2)) \\
&+ \mathcal{C}_{p(x_1)p(x_2)}[I(X_2;T_2|X_1,T_1) - I(X_2;Y_1|X_1,T_1)] \\
&- (I(X_2;T_2|X_1,T_1) - I(X_2;Y_1|X_1,T_1))\big),
\end{aligned}
$$

where $\mathcal{C}$ denotes the capacity region, and $\mathcal{C}_{p(x_1)p(x_2)}[\cdot]$ denotes the upper concave envelope evaluated with respect to the space of distributions $p(x_1)p(x_2)$.

Theorem 3.3 (Theorem 1 of [GN14]). *Let $\mathbf{Z}_1, \mathbf{Z}_2$ be independent n-dimensional Gaussian random variables and let $\lambda > 1$. Suppose K is an $n \times n$ positive semidefinite matrix. Then the maximization over distributions on $(U, \mathbf{X})$, where $\mathbf{X}$ is in $\mathbb{R}^n$ and $(U, \mathbf{X}), \mathbf{Z}_1, \mathbf{Z}_2$ are mutually independent,*

$$
\max_{\substack{p(u|\mathbf{x})p(\mathbf{x}) \\ \mathrm{E}[\mathbf{X}\mathbf{X}^T] \preceq K}} \left(I(\mathbf{X}; \mathbf{X}+\mathbf{Z}_1|U) - \lambda I(\mathbf{X}; \mathbf{X}+\mathbf{Z}_2|U) \right),
$$

is attained by some Gaussian $\mathbf{X}$ and constant U.

Corollary 3.2. *Let $\mathbf{X}$ be a random variable in $\mathbb{R}^n$ and $\mathbf{Z}_1, \mathbf{Z}_2$ be n-dimensional Gaussian random variables such that $\mathbf{X}$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are mutually independent. Let $\lambda > 1$. Then*

$$
\mathcal{C}_{\mathbf{X}}\left[I(\mathbf{X};\mathbf{X}+\mathbf{Z}_1) - \lambda I(\mathbf{X};\mathbf{X}+\mathbf{Z}_2)\right] \le I(\mathbf{X}^*;\mathbf{X}^*+\mathbf{Z}_1|\mathbf{U}^*) - \lambda I(\mathbf{X}^*;\mathbf{X}^*+\mathbf{Z}_2|\mathbf{U}^*),
$$

where $\mathbf{X}^* = \mathbf{U}^* + \mathbf{V}^*$ with $\mathbf{U}^*, \mathbf{V}^*$ being independent n-dimensional zero-mean Gaussian random variables, and $\mathrm{E}[\mathbf{X}^*\mathbf{X}^{*T}] = \mathrm{E}[\mathbf{X}\mathbf{X}^T]$.

Theorem 3.4. *For a GIC with $a(1+b^2P_2) + b(1+a^2P_1) \le 1$, for any $\lambda \ge 1$ satisfying*

$$
\lambda \le 1 + \frac{(1-a(1+b^2P_2))^2 - b^2(1+a^2P_1)^2}{b^2((1+a^2P_1)^2 + P_2(1-a(1+b^2P_2)))},
$$

it holds that

$$
\sup_{(R_1,R_2)\in\mathcal{C}} (\lambda R_1 + R_2) = \frac{\lambda}{2}\log\left(1+\frac{P_1}{1+b^2P_2}\right) + \frac{1}{2}\log\left(1+\frac{P_2}{1+a^2P_1}\right),
$$

where $\mathcal{C}$ denotes the capacity region.

Proof. Consider the GIC

$$
\begin{aligned}
Y_1 &= X_1 + bX_2 + Z_1, \\
Y_2 &= X_2 + aX_1 + Z_2,
\end{aligned}
$$

with power constraints $\mathrm{E}[X_i^2] = P_i$ $(i = 1, 2)$. Let $\theta_1, \theta_2 \in \mathbb{R}$. We parameterize the Gaussian noise as

$$
Z_1 = Z_{11}\sin\theta_1 + Z_{12}\cos\theta_1,
$$

$$Z_2 = Z_{21}\sin\theta_2 + Z_{22}\cos\theta_2,$$

where $Z_{ij} \sim \mathcal{N}(0,1)$ $(i,j \in \{1,2\})$ are mutually independent standard scalar Gaussian random variables. Define the auxiliary receivers

$$T_1 := X_1 + \eta_1 Z_{11},$$
$$T_2 := X_2 + \eta_2 Z_{21},$$

and let

$$\hat{Y}_1 := aX_1 + Z_{22}\cos\theta_2,$$
$$\hat{Y}_2 := bX_2 + Z_{12}\cos\theta_1,$$

where η_1, η_2 are defined by

$$\eta_1 := \frac{1 + b^2 P_2}{\sin\theta_1},$$
$$\eta_2 := \frac{1 + a^2 P_1}{\sin\theta_2}.$$

For $i = 1, 2$, let X_i^* be a zero-mean scalar Gaussian random variable such that $\mathrm{E}[X_i^{*2}] = \mathrm{E}[X_i^2]$, and define $Y_i^*, T_i^*, \hat{Y}_i^*$ analogously as $Y_i, T_i, \hat{Y}_i$ by using X_1^*, X_2^* instead of X_1, X_2. Note that our choices of η_1 and η_2 ensures that $X_1^* \to Y_1^* \to T_1^*$ and $X_2^* \to Y_2^* \to T_2^*$ form Markov chains.

With our choice of T_1, T_2, Theorem 3.2 gives that for any $\lambda \geq 1$ and achievable rate pair (R_1, R_2) for the GIC we have

$$\begin{aligned}\lambda R_1 + R_2 \leq \sup_{p(x_1)p(x_2)} \big(&\lambda I(X_1;T_1,Y_1) + I(X_2;T_2,Y_2) \\ &+ \mathcal{C}_{X_1}[I(X_1;T_1) - I(X_1;\hat{Y}_1)] - (I(X_1;T_1) - I(X_1;\hat{Y}_1)) \\ &+ \mathcal{C}_{X_2}[I(X_2;T_2) - \lambda I(X_2;\hat{Y}_2)] - (I(X_2;T_2) - \lambda I(X_2;\hat{Y}_2))\big). \quad (3.5)\end{aligned}$$

If θ_1, θ_2 satisfy

$$a^2\eta_1^2 \leq \cos^2\theta_2, \quad (3.6)$$
$$b^2\eta_2^2 \leq \cos^2\theta_1, \quad (3.7)$$

or in other words $\hat{Y}_1, \hat{Y}_2$ are stochastically degraded versions of T_1, T_2, respectively, then the following holds:

- $I(X_1;T_1) - I(X_1;\hat{Y}_1)$ is a concave functional in $p(x_1)$.
- We have

$$\lambda I(X_1;T_1,Y_1) + I(X_2;T_2,Y_2) - (I(X_2;T_2) - \lambda I(X_2;\hat{Y}_2))$$

$$
\begin{aligned}
&= \lambda h(T_1, Y_1) + h(Y_2|T_2) - h(\hat{Y}_1) - \lambda \log(\eta_1 \cos\theta_1) \\
&= \lambda h(Y_1|T_1) + h(Y_2|T_2) + (\lambda - 1)h(T_1) + (h(T_1) - h(\hat{Y}_1)) - \lambda \log(\eta_1 \cos\theta_1) \\
&\overset{(a)}{\leq} \lambda h(Y_1^*|T_1^*) + h(Y_2^*|T_2^*) + (\lambda - 1)h(T_1^*) + (h(T_1^*) - h(\hat{Y}_1^*)) - \lambda \log(\eta_1 \cos\theta_1) \\
&= \lambda I(X_1^*; T_1^*, Y_1^*) + I(X_2^*; T_2^*, Y_2^*) - (I(X_2^*; T_2^*) - \lambda I(X_2^*; \hat{Y}_2^*)) \\
&\overset{(b)}{=} \lambda I(X_1^*; Y_1^*) + I(X_2^*; Y_2^*) - (I(X_2^*; T_2^*) - \lambda I(X_2^*; \hat{Y}_2^*)),
\end{aligned}
$$

where (a) follows from the maximum conditional differential entropy property of Gaussians, and we have used (3.6) for the term $h(T_1) - h(\hat{Y}_1)$, and (b) follows from the Markov chains $X_1^* \to Y_1^* \to T_1^*$ and $X_2^* \to Y_2^* \to T_2^*$.

- By Corollary 3.2 we have

$$
\mathcal{C}_{X_2}[I(X_2; T_2) - \lambda I(X_2; \hat{Y}_2)] \leq I(X_2^*; T_2^*|U_2^*) - \lambda I(X_2^*; \hat{Y}_2^*|U_2^*)
$$

  for some independent scalar Gaussian random variables U_2^*, V_2^* such that $X_2^* = U_2^* + V_2^*$.

Hence (3.5) reduces to

$$
\lambda R_1 + R_2 \leq \lambda I(X_1; Y_1) + I(X_2; Y_2) + \max_{0 \leq \alpha \leq 1} \left(\lambda I(U_2; \hat{Y}_2) - I(U_2; T_2) \right),
$$

where $U_2 \sim \mathcal{N}(0, \alpha P_2)$, $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 = U_2 + V_2$ with $V_2 \sim \mathcal{N}(0, (1 - \alpha)P_2)$ independent of U_2. The maximization is attained by $\alpha = 0$ (i.e., U_2 is trivial) if

$$
\frac{\cos^2\theta_1 - \lambda b^2 \eta_2^2}{b^2 P_2(\lambda - 1)} \geq 1. \tag{3.8}
$$

Note that (3.8) automatically implies (3.7).

Now we choose $\theta_1 + \theta_2 = \frac{\pi}{2}$. For $\lambda \geq 1$ to satisfy (3.6), (3.7) and (3.8) it suffices that

$$
\begin{aligned}
a(1 + b^2 P_2) &\leq \sin^2\theta_1, \\
b^2 P_2(\lambda - 1)\cos^2\theta_1 &\leq \cos^4\theta_1 - \lambda b^2(1 + a^2 P_1)^2.
\end{aligned}
$$

After some computation one can show that there exists θ_1 satisfying these conditions if

$$
\begin{aligned}
\lambda &\leq \max_{\substack{\theta_1 \in \mathbb{R} \\ a(1+b^2P_2) \leq \sin^2\theta_1}} \left(1 + \frac{\cos^4\theta_1 - b^2(1 + a^2 P_1)^2}{b^2 P_2 \cos^2\theta_1 + b^2(1 + a^2 P_1)^2} \right) \\
&= 1 + \frac{(1 - a(1 + b^2 P_2))^2 - b^2(1 + a^2 P_1)^2}{b^2((1 + a^2 P_1)^2 + P_2(1 - a(1 + b^2 P_2)))}.
\end{aligned}
$$

This concludes the proof. □

Chapter 4

A Gaussian extremality conjecture

In this chapter we propose the following Conjecture 4.1, which, if true, would imply that the Han–Kobayashi achievable region for GZIC is attained by Gaussian inputs (conditioned on the time-sharing variable Q), i.e., $\mathcal{R}_n^{\text{HK}} = \mathcal{R}_n^{\text{HK-GS}}$, and in turn, together with Theorem 2.1, would imply that the single-letter Han–Kobayashi achievable region with Gaussian inputs $\mathcal{R}_1^{\text{HK-GS}}$ is a computable characterization of the capacity region of GZIC.

Conjecture 4.1. *Let $\lambda \geq 1$, $N_2 \geq 0$, and let $\Sigma_1, A_2 \succeq 0$ be $n \times n$ matrices ($n \geq 1$). The maximum*

$$\max_{\substack{p(\mathbf{x}_1)p(\mathbf{x}_2)\\ \mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T]\preceq A_2}} ((\lambda-1)h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2)+h(\mathbf{X}_1+\mathbf{Z}_1)-\lambda h(\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2)$$
$$-\operatorname{tr}(\Sigma_1\,\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T])),$$

where $\mathbf{Z}_1 \sim \mathcal{N}(0, I)$, $\mathbf{Z}_2 \sim \mathcal{N}(0, N_2 I)$ and $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2$ are random variables in $\mathbb{R}^n$, is attained by Gaussian $\mathbf{X}_1$ and $\mathbf{X}_2$.

This chapter has two main results:

(i) We show Theorem 4.1, which concerns a certain property that a matrix functional has, via information-theoretic methods. The significance of this result is that it bridges the gap between Conjecture 4.1 and the optimality of Gaussian inputs for Han–Kobayashi achievable region of GZIC, as shown in Proposition 4.3. This result first appeared in [Cos+20].

(ii) We then show an information inequality that establishes Conjecture 4.1 in some regimes. This inequality also yields an outer bound for the slope at the Costa–Polyanskiy–Wu corner point of capacity region of GZIC. This result first appeared in [GNN21].

Throughout this chapter we focus on the GZIC with weak interference ($0 < a \leq 1$). We consider an equivalent formulation [Cos85b] of the GZIC:

$$Y_1 = X_1 + Z_1, \tag{4.1a}$$

$$Y_2 = X_2 + X_1 + Z_1 + Z_2, \tag{4.1b}$$

where $Z_1 \sim \mathcal{N}(0,1)$, $Z_2 \sim \mathcal{N}(0, N_2)$ (with $N_2 := \frac{1}{a^2} - 1$ and $0 < a < 1$) and X_i, Y_i, Z_i ($i = 1, 2$) are random variables in $\mathbb{R}$, under the power constraints P_1 on X_1 and $\frac{1}{a^2}P_2$ on X_2. We also use the notations $\mathcal{R}_n^{\text{HK}}(Q_1, Q_2)$ (respectively, $\mathcal{R}_n^{\text{HK-GS}}(Q_1, Q_2)$) to denote the n-letter extension of the Han–Kobayashi achievable region (respectively, the region with Gaussian inputs) of this Z-interference channel with expected average power constraints Q_1 on X_1 and Q_2 on X_2, where $Q_1, Q_2 \geq 0$.

4.1 Prelimiaries

Upper concave envelope

Definition 4.1. *The* ***upper concave envelope*** $\mathcal{C}_x[f(x)]$ *of a functional f defined on some locally convex Hausdorff real topological vector space is defined by one of the many equivalent ways (cf. [Nai13]):*

(i) $\mathcal{C}_x[f(x)] := \inf\{g(x) : g$ *is a concave functional with* $g \geq f\}$,

(ii) $\mathcal{C}_x[f(x)] := \sup\{\int f \, d\mu : \mu$ *is a Borel probability measure with mean* $x\}$,

(iii) $\mathcal{C}_x[f(x)] := \inf_\alpha \left(\sup_{\hat{x}} \left(f(\hat{x}) - \langle \alpha, \hat{x} \rangle\right) + \langle \alpha, x \rangle\right)$, *where the infimum is over all linear functionals α, and $\langle \alpha, x \rangle := \alpha(x)$ denotes the evaluation of α at a point x.*

Proof for equivalence of the definitions (i),(ii),(iii). Let $\varphi_1, \varphi_2, \varphi_3$ denote the functionals in (i), (ii), (iii), respectively.

$\varphi_1 \leq \varphi_2$: By taking μ to be the Dirac measure one can see $\varphi_2 \geq f$. It remains to show concavity of φ_2. Let μ_1, μ_2 be Borel probability measures with means x_1, x_2, respectively, and let $0 \leq t \leq 1$. Then $\mu := t\mu_1 + (1-t)\mu_2$ is a Borel probability measure with mean $tx_1 + (1-t)x_2$. Hence

$$t \int f \, d\mu_1 + (1-t) \int f \, d\mu_2 = \int f \, d\mu \leq \varphi_2(tx_1 + (1-t)x_2).$$

Taking supremum over μ_1, μ_2 yields the result.

$\varphi_1 \geq \varphi_2$: Let g be a concave functional with $g \geq f$ and μ be a Borel probability measure with mean x. Jensen's inequality gives $g(x) \geq \int g \, d\mu \geq \int f \, d\mu$.

$\varphi_1 \leq \varphi_3$: For any linear functional α the function $x \mapsto \sup_{\hat{x}} (f(\hat{x}) - \langle \alpha, \hat{x} \rangle) + \langle \alpha, x \rangle$ bounds f from above and is affine (hence concave).

$\varphi_1 \geq \varphi_3$: It suffices that

$$g(x) \geq \inf_{\alpha} \left(\sup_{\hat{x}} (g(\hat{x}) - \langle \alpha, \hat{x} \rangle) + \langle \alpha, x \rangle \right)$$

for any x and concave functional g. This is an immediate consequence of the Hahn–Banach separation theorem (e.g. Theorem 1' of [Las96]).

□

A few lemmas

We state a few lemmas useful for proving Gaussian extremality.

- Lemma 4.1 is a well-known property called double Markovity (cf. Problem 16.25 of [CK11]).
- Lemma 4.2 relies on the fact that the characteristic function of a Gaussian random variable vanishes nowhere.
- Lemma 4.3 follows from a characterization of Gaussian random variables given by Ghurye and Olkin [GO62], which was shown by solving a functional equation, generalizing an earlier functional equation of Skitovich [Ski54], satisfied by the characteristic functions.
- Lemma 4.4 is a form of the celebrated Prokhorov theorem in measure theory which says that tight collections of distributions on $\mathbb{R}^d$ are weakly sequentially compact.
- Lemma 4.5 establishes weak continuity of the entropy functional under additive Gaussian noise.
- Lemma 4.6 establishes weak continuity of upper concave envelope of weakly continuous functional, under a bounded moment condition.
- Lemma 4.4, 4.5 and 4.6 are used to justify the existence of optimizing distributions for a class of optimization problems involving information functionals, through standard techniques in Appendix II of [GN14]: The covariance constraints gives tightness of measures and hence Prokhorov's theorem (Lemma 4.4) guarantees the existence of a weakly convergent sequence of distributions whose value of the objective functional tends to the maximum,

and Lemma 4.5 and 4.6 can be applied for our objective functional to establish its weak continuity, implying the limiting distribution attains the maximum. In particular, in our proof, the lemmas ensure the existence of maximizer in Proposition 4.1 and 4.5 (iii).

Lemma 4.1 (Double Markovity). *Let Q be a random variable and let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be random variables in $\mathbb{R}^n$ such that for any q the conditional distribution $p(\mathbf{x}, \mathbf{y}, \mathbf{z}|q)$ has everywhere non-zero density. Suppose*

$$\mathbf{X} \to (\mathbf{Y}, Q) \to \mathbf{Z} \quad \textit{and} \quad \mathbf{Y} \to (\mathbf{X}, Q) \to \mathbf{Z}$$

form Markov chains. Then

$$(\mathbf{X}, \mathbf{Y}) \to Q \to \mathbf{Z}$$

forms a Markov chain.

Proof. For any $q, \mathbf{x}, \mathbf{y}, \mathbf{z}$, the given Markov chains imply

$$p(\mathbf{z}|q, \mathbf{x}) = p(\mathbf{z}|q, \mathbf{x}, \mathbf{y}) = p(\mathbf{z}|q, \mathbf{y})$$

and hence

$$\begin{aligned} p(\mathbf{z}|q) &= \mathrm{E}_{\mathbf{X}}[p(\mathbf{z}|q, \mathbf{X})|Q = q] \\ &= \mathrm{E}_{\mathbf{X}}[p(\mathbf{z}|q, \mathbf{y})|Q = q] \\ &= p(\mathbf{z}|q, \mathbf{y}) \\ &= p(\mathbf{z}|q, \mathbf{x}, \mathbf{y}) \end{aligned}$$

as required. □

Lemma 4.2 (Proposition 2 of [GN14]). *Let $\mathbf{X}_1, \mathbf{X}_2$ be random variables in $\mathbb{R}^n$ and $\mathbf{Z}_1, \mathbf{Z}_2$ be n-dimensional Gaussian random variables such that $(\mathbf{X}_1, \mathbf{X}_2)$, $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are mutually independent. Suppose $\mathbf{X}_1 + \mathbf{Z}_1$ and $\mathbf{X}_2 + \mathbf{Z}_2$ are independent. Then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent.*

Lemma 4.3 (Corollary 3 of [GN14]). *Let $\mathbf{X}_1, \mathbf{X}_2$ be independent random variables in $\mathbb{R}^n$ such that $\mathbf{X}_1 + \mathbf{X}_2$ and $\mathbf{X}_1 - \mathbf{X}_2$ are independent. Then $\mathbf{X}_1, \mathbf{X}_2$ are Gaussians having the same covariance matrix.*

Lemma 4.4 (Prokhorov's, Theorem 4 of [GN14]). *Let $\{\mathbf{X}_n\}$ be a sequence of random variables in $\mathbb{R}^d$ ($d \geq 1$). Suppose the sequence $\{\mathbf{X}_n\}$ is tight, i.e., for any $\epsilon > 0$ there exists a compact subset K_ϵ of $\mathbb{R}^d$ such that $\mathrm{P}(\mathbf{X}_n \in K_\epsilon) \leq \epsilon$ for any n. Then $\{\mathbf{X}_n\}$ has a weakly convergent subsequence.*

Lemma 4.5 (Proposition 18 of [GN14]). *Let $\{\mathbf{X}_n\}$ be a sequence of random variables in $\mathbb{R}^d$ ($d \geq 1$) converging weakly to $\mathbf{X}^*$. Let $\mathbf{Z} \sim \mathcal{N}(0, I)$ be a d-dimensional Gaussian random variable independent of each $\mathbf{X}_n$ and $\mathbf{X}^*$. Suppose there exists $K \succeq 0$ such that $\mathrm{E}[\mathbf{X}^* \mathbf{X}^{*T}] \preceq K$ and $\mathrm{E}[\mathbf{X}_n \mathbf{X}_n^T] \preceq K$ for all n. Then*

(i) The distribution of $\mathbf{X}_n + \mathbf{Z}$ weakly converges to that of $\mathbf{X}^* + \mathbf{Z}$.

(ii) The density of $\mathbf{X}_n + \mathbf{Z}$ converges pointwise to that of $\mathbf{X}^* + \mathbf{Z}$.

(iii) $h(\mathbf{X}_n + \mathbf{Z})$ converges to $h(\mathbf{X}^* + \mathbf{Z})$.

Lemma 4.6 (Proposition 21 of [GN14]). *Let f be a real-valued functional defined on the set of all Borel probability distributions on $\mathbb{R}^d$ ($d \geq 1$). Suppose f is bounded and has the following property, P: for any sequence of distributions $\{\mathbf{X}_n\}$ converging weakly to $\mathbf{X}^*$ and satisfying $\inf_{\kappa>1} \sup_n \mathrm{E}[\|\mathbf{X}_n\|^\kappa] < \infty$ we have that $f(\mathbf{X}_n)$ converges to $f(\mathbf{X}^*)$. Then the upper concave envelope of f is also bounded and has property P.*

4.2 Conjecture 4.1 implies capacity region of GZIC

In this section we show Theorem 4.1, which constitutes a crucial step in the proof of Proposition 4.3 which establishes that Conjecture 4.1 implies that for GZIC the Han–Kobayashi achievable region with Gaussian inputs matches the capacity region.

Theorem 4.1. *Let $\lambda \geq 1$ and $N_2 \geq 0$. The functional ψ_G defined by*

$$\begin{aligned} \psi_G(K_1, K_2) := \frac{\lambda - 1}{2} &\log |K_2 + K_1 + I + N_2 I| \\ &+ \frac{1}{2} \log |K_1 + I| - \frac{\lambda}{2} \log |K_1 + I + N_2 I|, \end{aligned} \tag{4.2}$$

where $K_1, K_2 \succeq 0$ are $n \times n$ matrices ($n \geq 1$), satisfies

$$\mathcal{C}_{K_1}\left[\psi_G(K_1, K_2)\right] = \max_{\substack{\hat{K}_1 \succeq 0 \\ \hat{K}_1 \preceq K_1}} \psi_G(\hat{K}_1, K_2)$$

for any $K_1, K_2 \succeq 0$.

Remark 4.1. *For the scalar case Theorem 4.1 can be shown directly without much effort by showing the following properties of the function $\psi_G(Q_1, Q_2)$ for scalar $Q_1, Q_2 \geq 0$.*

(i) $Q_1 \mapsto \psi_G(Q_1, Q_2)$ either is increasing on $[0, +\infty)$, is decreasing on $[0, +\infty)$, or is increasing on $[0, Q_1^*)$ and decreasing on $[Q_1^*, +\infty)$ for some $Q_1^* > 0$,

(ii) $\partial_{Q_1}\psi_G(Q_1,Q_2)\geq 0$ *implies* $\partial^2_{Q_1}\psi_G(Q_1,Q_2)\leq 0$,

(iii) $\lim_{Q_1\to+\infty}\psi_G(Q_1,Q_2)=0$.

However, for high-dimensional spaces it does not seem to admit a simple extension of the above argument especially since K_1, K_2 *may not be simultaneously diagonalizable. This necessitated us to come up with the different argument. This technique adds to the list of linear algebra inequalities that have information-theoretic proofs [DCT91].*

Proof of Theorem 4.1. We first show the "$\leq$" side. For independent random variables $\mathbf{X}_1,\mathbf{X}_2$ in $\mathbb{R}^n$ we denote

$$\begin{aligned}\psi(\mathbf{X}_1,\mathbf{X}_2) := (\lambda-1)&h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2)\\ &+h(\mathbf{X}_1+\mathbf{Z}_1)-\lambda h(\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2),\end{aligned} \tag{4.3}$$

where $\mathbf{Z}_1\sim\mathcal{N}(0,I)$ and $\mathbf{Z}_2\sim\mathcal{N}(0,N_2I)$ are independent with $(\mathbf{X}_1,\mathbf{X}_2)$. Notice that when $\mathbf{X}_i\sim\mathcal{N}(0,K_i)$ for $i=1,2$ one has $\psi(\mathbf{X}_1,\mathbf{X}_2)=\psi_{\mathrm{G}}(K_1,K_2)$. Now fix $K_2\succeq 0$ and $\mathbf{X}_2\sim\mathcal{N}(0,K_2)$, and we have

$$\begin{aligned}\mathcal{C}_{K_1}\left[\psi_{\mathrm{G}}(K_1,K_2)\right] &\overset{\text{(a)}}{\leq} \max_{\substack{p(\mathbf{x}_1)p(u_1|\mathbf{x}_1)\\ \mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T]\preceq K_1}} \mathrm{E}_{U_1}[\psi(\mathbf{X}_1|_{U_1},\mathbf{X}_2)]\\ &\overset{\text{(b)}}{=} \max_{\substack{\hat{K}_1\succeq 0\\ \hat{K}_1\preceq K_1}} \psi_{\mathrm{G}}(\hat{K}_1,K_2)\end{aligned}$$

for any $K_1\succeq 0$, where (a) holds since the right-hand side is concave in K_1 and upper bounds $\psi_{\mathrm{G}}(K_1,K_2)$ (by taking U_1 to be constant and $\mathbf{X}_1\sim\mathcal{N}(0,K_1)$), and (b) follows from Proposition 4.1.

Now we show the "$\geq$" side. Using the dual characterization of upper concave envelope we get

$$\begin{aligned}\mathcal{C}_{K_1}\left[\psi_{\mathrm{G}}(K_1,K_2)\right] &= \inf_{\substack{\Sigma_1\\ \Sigma_1=\Sigma_1^T}}\left(\sup_{\hat{K}_1\succeq 0}\left(\psi_{\mathrm{G}}(\hat{K}_1,K_2)-\mathrm{tr}(\Sigma_1\hat{K}_1)\right)+\mathrm{tr}(\Sigma_1K_1)\right)\\ &\overset{\text{(a)}}{=} \inf_{\Sigma_1\succeq 0}\left(\sup_{\hat{K}_1\succeq 0}\left(\psi_{\mathrm{G}}(\hat{K}_1,K_2)-\mathrm{tr}(\Sigma_1\hat{K}_1)\right)+\mathrm{tr}(\Sigma_1K_1)\right)\\ &\geq \sup_{\hat{K}_1\succeq 0}\inf_{\Sigma_1\succeq 0}\left(\psi_{\mathrm{G}}(\hat{K}_1,K_2)-\mathrm{tr}(\Sigma_1\hat{K}_1)+\mathrm{tr}(\Sigma_1K_1)\right)\\ &\geq \max_{\substack{\hat{K}_1\succeq 0\\ \hat{K}_1\preceq K_1}}\psi_{\mathrm{G}}(\hat{K}_1,K_2)\end{aligned}$$

for $K_1,K_2\succeq 0$, where (a) follows from the fact that

$$\sup_{\hat{K}_1\succeq 0}\left(\psi_{\mathrm{G}}(\hat{K}_1,K_2)-\mathrm{tr}(\Sigma_1\hat{K}_1)\right)=+\infty$$

for any symmetric Σ_1 with $\Sigma_1\not\succeq 0$. $\square$

Proposition 4.1. *Let $\lambda \geq 1$, let $K_1 \succeq 0$ be an $n \times n$ matrix and let $\mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2$ be independent Gaussian random variables in $\mathbb{R}^n$ $(n \geq 1)$. Then the maximum*

$$\max_{\substack{p(\mathbf{x}_1)p(u_1|\mathbf{x}_1) \\ \mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T] \preceq K_1}} ((\lambda - 1)h(\mathbf{X}_2 + \mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2|U_1) \\ + h(\mathbf{X}_1 + \mathbf{Z}_1|U_1) - \lambda h(\mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2|U_1))$$

is attained by some zero-mean Gaussian $\mathbf{X}_1$ and constant random variable U_1.

Proof. By the translation-invariance of entropy we can without loss of generality assume $\mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2$ are zero-mean Gaussians. For any distribution $p(\mathbf{x}_1)$ on $\mathbb{R}^n$ denote

$$\psi(p(\mathbf{x}_1)) := (\lambda - 1)h(\mathbf{X}_2 + \mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2) + h(\mathbf{X}_1 + \mathbf{Z}_1) - \lambda h(\mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2)$$

where $\mathbf{X}_1 \sim p(\mathbf{x}_1)$. Let $p^*(\mathbf{x}_1, u_1)$ be a maximizer (existence of which can be justified using Lemma 4.4, 4.5 and 4.6 through techniques in Appendix II of [GN14]) for

$$v := \max_{\substack{p(\mathbf{x}_1)p(u_1|\mathbf{x}_1) \\ \mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T] \preceq K_1}} \mathrm{E}_{U_1}[\psi(p(\mathbf{x}_1|U_1))].$$

Assume without loss of generality that $p^*(\mathbf{x}_1|u_1)$ has mean zero, or otherwise replace $\mathbf{X}_1$ by $\mathbf{X}_1 - \mathrm{E}[\mathbf{X}_1|U_1]$, which indeed satisfies the constraint. To prove our proposition it suffices to show that $p^*(\mathbf{x}_1|u_1)$ is a Gaussian distribution with covariance matrix independent of choice of u_1. We shall show this by a subadditivity argument.

Define the random variables

$$(\mathbf{X}_{11}^*, U_{11}^*, \mathbf{X}_{12}^*, U_{12}^*) \sim p^*(\mathbf{x}_{11}^*, u_{11}^*)p^*(\mathbf{x}_{12}^*, u_{12}^*),$$

as well as

$$\mathbf{X}_{11} := \frac{\mathbf{X}_{11}^* + \mathbf{X}_{12}^*}{\sqrt{2}}, \qquad \mathbf{X}_{12} := \frac{\mathbf{X}_{11}^* - \mathbf{X}_{12}^*}{\sqrt{2}},$$

and $U_1 := (U_{11}^*, U_{12}^*)$. For $i = 1, 2$ denote

$$\begin{aligned} \mathbf{Y}_{1i} &:= \mathbf{X}_{1i} + \mathbf{Z}_{1i}, \\ \mathbf{Y}_{2i} &:= \mathbf{X}_{1i} + \mathbf{Z}_{1i} + \mathbf{Z}_{2i}, \\ \mathbf{Y}_{3i} &:= \mathbf{X}_{1i} + \mathbf{Z}_{1i} + \mathbf{Z}_{2i} + \mathbf{X}_{2i}, \end{aligned}$$

where $(\mathbf{X}_{2i}, \mathbf{Z}_{1i}, \mathbf{Z}_{2i})$ are identically distributed with $(\mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2)$. We have

$$2v = \mathrm{E}_{U_{11}^*}[\psi(p(\mathbf{x}_{11}^*|U_{11}^*))] + \mathrm{E}_{U_{12}^*}[\psi(p(\mathbf{x}_{12}^*|U_{12}^*))]$$

$$
\begin{aligned}
&= \mathrm{E}_{U_1}[\psi(p(\mathbf{x}_{11}^*|U_1)) + \psi(p(\mathbf{x}_{12}^*|U_1))] \\
&\overset{(a)}{=} (\lambda-1)h(\mathbf{Y}_{31},\mathbf{Y}_{32}|U_1) + h(\mathbf{Y}_{11},\mathbf{Y}_{12}|U_1) - \lambda h(\mathbf{Y}_{21},\mathbf{Y}_{22}|U_1) \\
&= (\lambda-1)(h(\mathbf{Y}_{31}|\mathbf{Y}_{32},U_1) + h(\mathbf{Y}_{32}|\mathbf{Y}_{11},U_1) + I(\mathbf{Y}_{11};\mathbf{Y}_{32}|U_1)) \\
&\qquad + (h(\mathbf{Y}_{11}|\mathbf{Y}_{32},U_1) + h(\mathbf{Y}_{12}|\mathbf{Y}_{11},U_1) + I(\mathbf{Y}_{11};\mathbf{Y}_{32}|U_1)) \\
&\qquad - \lambda(h(\mathbf{Y}_{21}|\mathbf{Y}_{32},U_1) + h(\mathbf{Y}_{22}|\mathbf{Y}_{11},U_1) \\
&\qquad\qquad + I(\mathbf{Y}_{21};\mathbf{Y}_{32}|U_1) + I(\mathbf{Y}_{11};\mathbf{Y}_{22}|U_1) - I(\mathbf{Y}_{21};\mathbf{Y}_{22}|U_1)) \\
&= \mathrm{E}[\psi(p(\mathbf{x}_{11}|\mathbf{Y}_{32},U_1))] + \mathrm{E}[\psi(p(\mathbf{x}_{12}|\mathbf{Y}_{11},U_1))] \\
&\qquad + \lambda(I(\mathbf{Y}_{11};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{21};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{11};\mathbf{Y}_{22}|U_1) + I(\mathbf{Y}_{21};\mathbf{Y}_{22}|U_1)) \\
&\overset{(b)}{\leq} 2v - \lambda I(\mathbf{Y}_{11};\mathbf{Y}_{22}|\mathbf{Y}_{21},\mathbf{Y}_{32},U_1),
\end{aligned}
$$

where (a) can be shown by rotational invariance of entropy, and (b) follows from

$$
\begin{aligned}
& I(\mathbf{Y}_{11};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{21};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{11};\mathbf{Y}_{22}|U_1) + I(\mathbf{Y}_{21};\mathbf{Y}_{22}|U_1) \\
&\overset{(c)}{=} I(\mathbf{Y}_{11};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{21};\mathbf{Y}_{32}|U_1) - I(\mathbf{Y}_{11};\mathbf{Y}_{22},\mathbf{Y}_{32}|U_1) + I(\mathbf{Y}_{21};\mathbf{Y}_{22},\mathbf{Y}_{32}|U_1) \\
&= -I(\mathbf{Y}_{11};\mathbf{Y}_{22}|\mathbf{Y}_{32},U_1) + I(\mathbf{Y}_{21};\mathbf{Y}_{22}|\mathbf{Y}_{32},U_1) \\
&\overset{(d)}{=} -I(\mathbf{Y}_{11},\mathbf{Y}_{21};\mathbf{Y}_{22}|\mathbf{Y}_{32},U_1) + I(\mathbf{Y}_{21};\mathbf{Y}_{22}|\mathbf{Y}_{32},U_1) \\
&= -I(\mathbf{Y}_{11};\mathbf{Y}_{22}|\mathbf{Y}_{21},\mathbf{Y}_{32},U_1),
\end{aligned}
$$

where (c) holds since $\mathbf{Y}_{32} \to (\mathbf{Y}_{22}, U_1) \to \mathbf{Y}_{11}$ and $\mathbf{Y}_{32} \to (\mathbf{Y}_{22}, U_1) \to \mathbf{Y}_{21}$ form Markov chains, and (d) holds since $\mathbf{Y}_{21} \to (\mathbf{Y}_{11}, \mathbf{Y}_{32}, U_1) \to \mathbf{Y}_{22}$ forms a Markov chain. Hence we have $I(\mathbf{Y}_{11};\mathbf{Y}_{22}|\mathbf{Y}_{21},\mathbf{Y}_{32},U_1) = 0$ and so

$$\mathbf{Y}_{11} \to (\mathbf{Y}_{21}, \mathbf{Y}_{32}, U_1) \to \mathbf{Y}_{22}$$

forms a Markov chain. Since we also have the Markov chain

$$\mathbf{Y}_{21} \to (\mathbf{Y}_{11}, \mathbf{Y}_{32}, U_1) \to \mathbf{Y}_{22},$$

by Lemma 4.1 we obtain a Markov chain

$$(\mathbf{Y}_{11}, \mathbf{Y}_{21}) \to (\mathbf{Y}_{32}, U_1) \to \mathbf{Y}_{22}.$$

Again we also have the Markov chain

$$(\mathbf{Y}_{11}, \mathbf{Y}_{21}) \to (\mathbf{Y}_{22}, U_1) \to \mathbf{Y}_{32},$$

and hence by Lemma 4.1 we obtain a Markov chain

$$(\mathbf{Y}_{11}, \mathbf{Y}_{21}) \to U_1 \to (\mathbf{Y}_{22}, \mathbf{Y}_{32}).$$

Lemma 4.2 yields the Markov chain $\mathbf{X}_{11} \to U_1 \to \mathbf{X}_{12}$. Hence for any u_{11}^*, u_{12}^* we have that $\mathbf{X}_{11}^*|_{U_{11}^*=u_{11}^*} + \mathbf{X}_{12}^*|_{U_{12}^*=u_{12}^*}$ and $\mathbf{X}_{11}^*|_{U_{11}^*=u_{11}^*} - \mathbf{X}_{12}^*|_{U_{12}^*=u_{12}^*}$ are independent, which, by Lemma 4.3, in turn implies that $p(\mathbf{x}_{11}^*|u_{11}^*)$ and $p(\mathbf{x}_{12}^*|u_{12}^*)$ are Gaussian distributions having the same covariance matrix. Thus we can conclude that the maximizing distribution $(\mathbf{X}_1, U_1) \sim p^*(\mathbf{x}_1, u_1)$ must satisfy

$$\mathbf{X}_1|_{U_1=u_1} \sim \mathcal{N}(\mu_{u_1}, \hat{K}_1)$$

for some $\mu_{u_1} \in \mathbb{R}^n$ and $\hat{K}_1 \succeq 0$. Finally $\mu_{u_1} = 0$ since $p^*(\mathbf{x}_1|u_1)$ is zero-mean. $\square$

It can be shown that for $\lambda \geq 1$ and $Q_1, Q_2 \geq 0$ we have

$$\max_{(R_1,R_2)\in\mathcal{R}_n^{\mathrm{HK}}(Q_1,Q_2)} n(R_1+\lambda R_2) = \mathcal{C}_{Q_1,Q_2}\Big[\max_{\substack{p(\mathbf{x}_1)p(\mathbf{x}_2)\\ \mathrm{E}[\|\mathbf{X}_1\|^2]\leq nQ_1\\ \mathrm{E}[\|\mathbf{X}_2\|^2]\leq nQ_2}} f_\lambda(\mathbf{X}_1,\mathbf{X}_2)\Big], \tag{4.4}$$

$$\max_{(R_1,R_2)\in\mathcal{R}_n^{\text{HK-GS}}(Q_1,Q_2)} n(R_1+\lambda R_2) = \mathcal{C}_{Q_1,Q_2}\Big[\max_{\substack{K_1,K_2\succeq 0\\ \mathrm{tr}(K_1)\leq nQ_1\\ \mathrm{tr}(K_2)\leq nQ_2}} f_\lambda^{\mathrm{GS}}(K_1,K_2)\Big], \tag{4.5}$$

where

$$f_\lambda(\mathbf{X}_1,\mathbf{X}_2) := h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2) - h(\mathbf{Z}_1) + \mathcal{C}_{\mathbf{X}_1}\big[\psi(\mathbf{X}_1,\mathbf{X}_2)\big], \tag{4.6}$$

$$f_\lambda^{\mathrm{GS}}(K_1,K_2) := \frac{1}{2}\log|K_2+K_1+I+N_2I| + \max_{\substack{\hat{K}_1\succeq 0\\ \hat{K}_1\preceq K_1}} \psi_{\mathrm{G}}(\hat{K}_1,K_2), \tag{4.7}$$

with ψ, ψ_{G} defined by (4.3), (4.2), respectively.

One of the main difficulties in making a Gaussian extremality conjecture directly for the expression in (4.6) is that previous work has shown that Gaussian signaling with non-trivial Q (or power control) can improve on Gaussian signaling with a constant Q. Hence, Conjecture 4.1 is obtained by utilizing a carefully constructed dual functional.

As the reader will see the main difficulty in proving $\mathcal{R}_n^{\mathrm{HK}}(P_1,P_2) = \mathcal{R}_n^{\text{HK-GS}}(P_1,P_2)$ is to establish the upper bound

$$\mathcal{C}_{\mathbf{X}_1}\big[\psi(\mathbf{X}_1,\mathbf{X}_2)\big] \leq \max_{\substack{\hat{K}_1\succeq 0\\ \hat{K}_1\preceq K_1}} \psi_{\mathrm{G}}(\hat{K}_1,K_2)$$

with $K_i = \mathrm{E}[\mathbf{X}_i\mathbf{X}_i^T]$ $(i=1,2)$. While Conjecture 4.1 implies

$$\mathcal{C}_{\mathbf{X}_1}\big[\psi(\mathbf{X}_1,\mathbf{X}_2)\big] \leq \mathcal{C}_{K_1}\big[\psi_{\mathrm{G}}(K_1,K_2)\big]$$

as one can see in the proof of Proposition 4.3, there is still a missing link as it is in general not true for all functionals ϕ that $\mathcal{C}_{K_1}[\phi(K_1)] = \max_{0\preceq\hat{K}_1\preceq K_1}\phi(\hat{K}_1)$. However Theorem 4.1 says that $K_1 \mapsto \psi_{\mathrm{G}}(K_1,K_2)$ has such property, constituting the key step towards Proposition 4.3.

Proposition 4.2. *Let $n \geq 1$. The following are equivalent:*

(i) For any $P_1, P_2 \geq 0$ it holds that

$$\mathcal{R}_n^{HK}(P_1,P_2) = \mathcal{R}_n^{HK\text{-}GS}(P_1,P_2).$$

(ii) For any $\lambda \geq 1$ and $\alpha_1, \alpha_2 \geq 0$ it holds that

$$\begin{aligned}&\sup_{p(\mathbf{x}_1)p(\mathbf{x}_2)} \big(f_\lambda(\mathbf{X}_1,\mathbf{X}_2) - \alpha_1 \operatorname{E}[\|\mathbf{X}_1\|^2] - \alpha_2 \operatorname{E}[\|\mathbf{X}_2\|^2]\big)\\ &\leq \sup_{K_1,K_2\succeq 0} \big(f_\lambda^{GS}(K_1,K_2) - \alpha_1 \operatorname{tr}(K_1) - \alpha_2 \operatorname{tr}(K_2)\big),\end{aligned}$$

where $\mathbf{X}_1, \mathbf{X}_2$ are in $\mathbb{R}^n$, K_1, K_2 are $n \times n$ matrices, and f_λ, f_λ^{GS} are defined by (4.6), (4.7), respectively.

Proof. By (4.4) one has that for $\lambda \geq 1$ and $P_1, P_2 \geq 0$,

$$\begin{aligned}&\max_{(R_1,R_2)\in\mathcal{R}_n^{\mathrm{HK}}(P_1,P_2)} n(R_1+\lambda R_2)\\ &= \inf_{\alpha_1,\alpha_2\in\mathbb{R}} \Bigg(\sup_{\substack{Q_1,Q_2\geq 0,\, p(\mathbf{x}_1)p(\mathbf{x}_2)\\ \operatorname{E}[\|\mathbf{X}_1\|^2]\leq nQ_1\\ \operatorname{E}[\|\mathbf{X}_2\|^2]\leq nQ_2}} \big(f_\lambda(\mathbf{X}_1,\mathbf{X}_2) - \alpha_1 nQ_1 - \alpha_2 nQ_2\big) + \alpha_1 nP_1 + \alpha_2 nP_2\Bigg)\\ &= \inf_{\alpha_1,\alpha_2\geq 0} \Bigg(\sup_{p(\mathbf{x}_1)p(\mathbf{x}_2)} \big(f_\lambda(\mathbf{X}_1,\mathbf{X}_2) - \alpha_1 \operatorname{E}[\|\mathbf{X}_1\|^2] - \alpha_2 \operatorname{E}[\|\mathbf{X}_2\|^2]\big) + \alpha_1 nP_1 + \alpha_2 nP_2\Bigg),\end{aligned}$$

and similarly by (4.5),

$$\begin{aligned}&\max_{(R_1,R_2)\in\mathcal{R}_n^{\mathrm{HK\text{-}GS}}(P_1,P_2)} n(R_1+\lambda R_2)\\ &= \inf_{\alpha_1,\alpha_2\geq 0} \Bigg(\sup_{K_1,K_2\succeq 0} \big(f_\lambda^{\mathrm{GS}}(K_1,K_2) - \alpha_1 \operatorname{tr}(K_1) - \alpha_2 \operatorname{tr}(K_2)\big) + \alpha_1 nP_1 + \alpha_2 nP_2\Bigg).\end{aligned}$$

Note that this already gives that (ii) implies (i). Now assuming (i) we have

$$\begin{aligned}&\sup_{K_1,K_2\succeq 0} \big(f_\lambda^{\mathrm{GS}}(K_1,K_2) - \alpha_1 \operatorname{tr}(K_1) - \alpha_2 \operatorname{tr}(K_2)\big)\\ &\geq \max_{(R_1,R_2)\in\mathcal{R}_n^{\mathrm{HK\text{-}GS}}(P_1,P_2)} n(R_1+\lambda R_2) - \alpha_1 nP_1 - \alpha_2 nP_2\\ &\geq \max_{\substack{p(\mathbf{x}_1)p(\mathbf{x}_2)\\ \operatorname{E}[\|\mathbf{X}_1\|^2]\leq nP_1\\ \operatorname{E}[\|\mathbf{X}_2\|^2]\leq nP_2}} f_\lambda(\mathbf{X}_1,\mathbf{X}_2) - \alpha_1 nP_1 - \alpha_2 nP_2\end{aligned}$$

for any $\lambda \geq 1$, $P_1, P_2 \geq 0$ and $\alpha_1, \alpha_2 \geq 0$. Finally, taking supremum over $P_1, P_2 \geq 0$ on the last step gives (ii). □

Proposition 4.3. *Conjecture 4.1 implies that*

$$\mathcal{R}_n^{HK}(P_1,P_2) = \mathcal{R}_n^{HK\text{-}GS}(P_1,P_2)$$

for any $n \geq 1$ and $P_1, P_2 \geq 0$.

Proof. We shall prove the proposition by showing that Conjecture 4.1 implies (ii) of Proposition 4.2.

Conjecture 4.1 implies that for any $\Sigma_1, A_2 \succeq 0$ we have

$$\begin{aligned}
&\sup_{\substack{p(\mathbf{x}_1)p(\mathbf{x}_2)\\ \mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T]\preceq A_2}} \Big(\psi(\mathbf{X}_1,\mathbf{X}_2) - \mathrm{tr}(\Sigma_1\,\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T])\Big)\\
&= \sup_{\substack{K_1,K_2\succeq 0,\,\mu_1,\mu_2\in\mathbb{R}^n\\ K_2+\mu_2\mu_2^T\preceq A_2}} \Big(\psi_{\mathrm{G}}(K_1,K_2) - \mathrm{tr}(\Sigma_1 K_1) - \mu_1^T\Sigma_1\mu_1\Big)\\
&= \sup_{\substack{K_1,K_2\succeq 0\\ K_2\preceq A_2}} \Big(\psi_{\mathrm{G}}(K_1,K_2) - \mathrm{tr}(\Sigma_1 K_1)\Big)\\
&= \sup_{K_1\succeq 0} \Big(\psi_{\mathrm{G}}(K_1,A_2) - \mathrm{tr}(\Sigma_1 K_1)\Big),
\end{aligned} \tag{4.8}$$

where the last equality is a consequence of the monotonicity of $\psi_{\mathrm{G}}(K_1,K_2)$ in K_2. Then for any $p(\mathbf{x}_1)p(\mathbf{x}_2)$ it holds that

$$\begin{aligned}
&\mathcal{C}_{\mathbf{X}_1}\big[\psi(\mathbf{X}_1,\mathbf{X}_2)\big]\\
&\overset{(a)}{\le} \inf_{\Sigma_1\succeq 0}\Big(\sup_{p(\hat{\mathbf{x}}_1)}\big(\psi(\hat{\mathbf{X}}_1,\mathbf{X}_2) - \mathrm{E}[\hat{\mathbf{X}}_1^T\Sigma_1\hat{\mathbf{X}}_1]\big) + \mathrm{E}[\mathbf{X}_1^T\Sigma_1\mathbf{X}_1]\Big)\\
&\le \inf_{\Sigma_1\succeq 0}\Big(\sup_{\substack{p(\hat{\mathbf{x}}_1)p(\hat{\mathbf{x}}_2)\\ \mathrm{E}[\hat{\mathbf{X}}_2\hat{\mathbf{X}}_2^T]\preceq\mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T]}}\big(\psi(\hat{\mathbf{X}}_1,\hat{\mathbf{X}}_2) - \mathrm{tr}(\Sigma_1\,\mathrm{E}[\hat{\mathbf{X}}_1\hat{\mathbf{X}}_1^T])\big) + \mathrm{tr}(\Sigma_1\,\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T])\Big)\\
&\overset{(b)}{=} \inf_{\Sigma_1\succeq 0}\Big(\sup_{K_1\succeq 0}\Big(\psi_{\mathrm{G}}(K_1,\mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T]) - \mathrm{tr}(\Sigma_1K_1)\Big) + \mathrm{tr}(\Sigma_1\,\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T])\Big)\\
&\overset{(c)}{=} \inf_{\substack{\Sigma_1\\ \Sigma_1=\Sigma_1^T}}\Big(\sup_{K_1\succeq 0}\Big(\psi_{\mathrm{G}}(K_1,\mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T]) - \mathrm{tr}(\Sigma_1K_1)\Big) + \mathrm{tr}(\Sigma_1\,\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T])\Big)\\
&\overset{(d)}{=} \mathcal{C}_{K_1}\big[\psi_{\mathrm{G}}(K_1,\mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T])\big]\Big|_{K_1=\mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T]},
\end{aligned}$$

where (a) holds since the right-hand side is a concave functional in $p(\mathbf{x})$ that upper bounds $\psi(\mathbf{X}_1,\mathbf{X}_2)$, (b) follows from (4.8), (c) holds since the inner supremum equals $+\infty$ for any symmetric Σ_1 with $\Sigma_1 \not\succeq 0$, and (d) is the dual characterization of upper concave envelope. Now note also that for any $p(\mathbf{x}_1)p(\mathbf{x}_2)$,

$$h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2) - h(\mathbf{Z}_1) \le \frac{1}{2}\log|\,\mathrm{E}[\mathbf{X}_2\mathbf{X}_2^T] + \mathrm{E}[\mathbf{X}_1\mathbf{X}_1^T] + I + N_2 I|$$

and hence for any $\lambda \ge 1$ and $\alpha_1,\alpha_2 \ge 0$ we have

$$\begin{aligned}
&\sup_{p(\mathbf{x}_1)p(\mathbf{x}_2)}\big(f_\lambda(\mathbf{X}_1,\mathbf{X}_2) - \alpha_1\,\mathrm{E}[\|\mathbf{X}_1\|^2] - \alpha_2\,\mathrm{E}[\|\mathbf{X}_2\|^2]\big)\\
&\le \sup_{K_1,K_2\succeq 0}\Big(\frac{1}{2}\log|K_2+K_1+I+N_2I| + \mathcal{C}_{K_1}\big[\psi_{\mathrm{G}}(K_1,K_2)\big] - \alpha_1\,\mathrm{tr}(K_1) - \alpha_2\,\mathrm{tr}(K_2)\Big)\\
&\overset{(e)}{=} \sup_{K_1,K_2\succeq 0}\big(f_\lambda^{\mathrm{GS}}(K_1,K_2) - \alpha_1\,\mathrm{tr}(K_1) - \alpha_2\,\mathrm{tr}(K_2)\big)
\end{aligned}$$

as required, where (e) follows from (4.7) and Theorem 4.1. $\square$

4.3 Proof of Conjecture 4.1 in some regimes

In this section we present a proof of the following Theorem 4.2 which is an information inequality that implies Conjecture 4.1 for some choices of parameters.

Theorem 4.2. *Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Z}_1, \mathbf{Z}_2$ be mutually independent random variables in $\mathbb{R}^n$ $(n \geq 1)$ with $\mathbf{Z}_1 \sim \mathcal{N}(0, N_1 I)$ and $\mathbf{Z}_2 \sim \mathcal{N}(0, N_2 I)$, where $N_1, N_2 > 0$. Suppose*

$$\mathrm{E}[\mathbf{X}_1] = \mathrm{E}[\mathbf{X}_2] = 0,$$
$$\mathrm{E}[\|\mathbf{X}_1\|^2],\, \mathrm{E}[\|\mathbf{X}_2\|^2] < \infty.$$

Then for any $\lambda \geq \lambda_1$, where

$$\lambda_1 := 1 + \frac{N_2}{N_1} \frac{1}{\left(1 - \sqrt{\frac{N_1+N_2}{\frac{1}{n}\mathrm{E}[\|\mathbf{X}_2\|^2]+N_1+N_2}}\right)^2},$$

we have

$$\begin{aligned}&(\lambda-1)h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2) + h(\mathbf{X}_1+\mathbf{Z}_1) - \lambda h(\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2)\\ &\leq \frac{n}{2}\left((\lambda-1)\log\left(\frac{1}{n}\mathrm{E}[\|\mathbf{X}_2\|^2] + N_1 + N_2\right) + \log N_1 - \lambda\log(N_1+N_2)\right).\end{aligned} \tag{4.9}$$

Remark 4.2. *The following points are worth noting:*

(i) The inequality is tight when $\mathbf{X}_1 = 0$ and $\mathbf{X}_2 \sim \mathcal{N}(0, P_2 I)$, for any $P_2 \geq 0$. Hence it establishes Conjecture 4.1 for $\lambda \geq \lambda_1$, $A_2 = P_2 I$ and any choice of $\Sigma_1 \succeq 0$.

(ii) The inequality does not hold for

$$\lambda < 1 + \frac{N_2}{N_1}\frac{1}{\left(1 + \frac{N_1+N_2}{\frac{1}{n}\mathrm{E}[\|\mathbf{X}_2\|^2]}\right)}.$$

This can be seen by setting $\mathbf{X}_2 \sim \mathcal{N}(0, P_2 I)$ and $\mathbf{X}_1 = \epsilon I$, and taking derivative of the left-hand side of the inequality with respect to ϵ at $\epsilon = 0$.

(iii) The inequality implies an outer bound to the slope of capacity region of GZIC around the Costa–Polyanskiy–Wu corner point, as shown in Proposition 4.4.

Proposition 4.4. *For a GZIC, for any $\lambda \geq \lambda_2$, where*

$$\lambda_2 := 1 + \frac{(1+P_2)(1-a^2)}{a^2 P_2}\frac{(1+\sqrt{1+P_2})^2}{P_2},$$

we have

$$\sup_{(R_1,R_2)\in\mathcal{C}} (R_1 + \lambda R_2) = \frac{1}{2}\log\left(1 + \frac{a^2 P_1}{1+P_2}\right) + \frac{\lambda}{2}\log(1+P_2),$$

where $\mathcal{C}$ denotes the capacity region.

Proof. We consider an equivalent formulation (4.1) of the GZIC. From Fano's inequality, any sequence of codebooks of rate (R_1, R_2), whose average probability of error goes to zero, must satisfy

$$\begin{aligned}
&R_1 + \lambda R_2 - \epsilon_n \\
&\leq \frac{1}{n}\left(I(X_1^n; Y_1^n) + \lambda I(X_2^n; Y_2^n)\right) \\
&= \frac{1}{n}\left(h(Y_2^n) - h(Y_1^n|X_1^n) + (\lambda-1)h(Y_2^n) + h(Y_1^n) - \lambda h(Y_2^n|X_2^n)\right) \\
&= \frac{1}{n}(h(X_2^n + X_1^n + Z_1^n + Z_2^n) - h(Z_1^n) \\
&\qquad + (\lambda-1)h(X_2^n + X_1^n + Z_1^n + Z_2^n) + h(X_1^n + Z_1^n) - \lambda h(X_1^n + Z_1^n + Z_2^n)) \\
&\overset{\text{(a)}}{\leq} \frac{1}{2}\left(\log\left(\frac{P_2}{a^2} + P_1 + \frac{1}{a^2}\right) + (\lambda-1)\log\left(\frac{P_2}{a^2} + \frac{1}{a^2}\right) - \lambda\log\left(\frac{1}{a^2}\right)\right) \\
&= \frac{1}{2}\log\left(1 + \frac{a^2 P_1}{1+P_2}\right) + \frac{\lambda}{2}\log(1+P_2),
\end{aligned}$$

where $\epsilon_n \to 0$, (a) follows from upper bounding $h(X_2^n + X_1^n + Z_1^n + Z_2^n)$ with the value of a Gaussian with the same power, and the latter terms using Theorem 4.2, where we use $\lambda \geq \lambda_2$. $\square$

Outline of the proof of Theorem 4.2

The proof of Theorem 4.2 follows the following steps.

(i) We upper bound the left-hand side of (4.9) by a functional $\Theta_{\lambda,\alpha}$ obtained by a relaxation of the optimization problem.

(ii) We show that $\Theta_{\lambda,\alpha}$ has Gaussian maximizers under certain covariance constraints. The proof employs the doubling technique developed by [GN14]. Instead of showing Gaussian optimality of $\Theta_{\lambda,\alpha}$ directly, we show that for a perturbed functional $\Theta_{\lambda,\alpha}^{(\epsilon,\delta)}$, which possesses a strict subadditive property as demonstrated in Lemma 4.7, giving the desired Markov chains for establishing Gaussian optimality. The Gaussian optimality of the unperturbed functional $\Theta_{\lambda,\alpha}$ then follows from uniform convergence as shown in Lemma 4.8.

Apart from Gaussian optimality, the doubling argument also gives the followings:

- $\Theta_{\lambda,\alpha}$ is additive at the Gaussian maximizer, as shown in Proposition 4.5 (i).
- The maximum value of $\Theta_{\lambda,\alpha}$ is concave in the power constraints, as shown in Proposition 4.5 (ii).

These properties are essential in the further reduction of the maximum value of $\Theta_{\lambda,\alpha}$.

(iii) We optimize $\Theta_{\lambda,\alpha}(\mathbf{X}_1, \mathbf{X}_2)$ over Gaussian distributions to show that, for large enough λ, the maximizer in given by $\mathbf{X}_2$ being Gaussian with full power and $\mathbf{X}_1$ being constant. Further for this choice the value of $\Theta_{\lambda,\alpha}(\mathbf{X}_1, \mathbf{X}_2)$ matches the left-hand side of (4.9).

At the end of the section we state all the lemmas required for the proof of Theorem 4.2.

Notation

Throughout the proof of Theorem 4.2 we shall use the following notations unless otherwise specified. Let $\mathbf{X}_{1i}$, $\mathbf{X}_{2i}$ $(i = 1, \ldots, n)$ be random variables in $\mathbb{R}^d$ $(d \geq 1)$ and let V, W be random variables in arbitrary alphabet sets. The joint distribution of $(\mathbf{X}_1^n, \mathbf{X}_2^n, V, W)$ is arbitrary. For $i = 1, \ldots, n$, let

$$\begin{aligned}
\mathbf{Y}_{1i} &:= \mathbf{X}_{1i} + \mathbf{Z}_{1i}, \\
\mathbf{T}_{2i} &:= \mathbf{X}_{2i} + \mathbf{Z}_{1i} + \mathbf{Z}_{2i}, \\
\mathbf{Y}_{2i} &:= \mathbf{X}_{2i} + \mathbf{X}_{1i} + \mathbf{Z}_{1i} + \mathbf{Z}_{2i} = \mathbf{T}_{2i} + \mathbf{X}_{1i},
\end{aligned}$$

where $\mathbf{Z}_{1i} \sim \mathcal{N}(0, N_1 I)$ and $\mathbf{Z}_{2i} \sim \mathcal{N}(0, N_2 I)$ are Gaussian random variables in $\mathbb{R}^d$ mutually independent of each other and of $(\mathbf{X}_1^n, \mathbf{X}_2^n, V, W)$. For any $\lambda \geq 1$ and $0 \leq \alpha \leq 1$, on the set of distributions $p(\mathbf{x}_1^n)p(\mathbf{x}_2^n)$ define the functional

$$\Theta_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n) := \sup_{\substack{p(v,w|\mathbf{x}_1^n,\mathbf{x}_2^n) \\ \mathbf{X}_1^n \to (V,W) \to \mathbf{X}_2^n \\ W \to V \to \mathbf{X}_2^n \\ \mathbf{X}_1^n \perp (V, \mathbf{X}_2^n)}} \theta_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W),$$

where

$$\begin{aligned}
&\theta_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W) \\
&:= (\lambda - 1)\alpha \cdot I(\mathbf{X}_2^n; \mathbf{T}_2^n | V) + (\lambda - 1)(1 - \alpha) \cdot (h(\mathbf{Y}_2^n | W) - h(\mathbf{Y}_1^n | W)) \\
&\quad + (1 + (\lambda - 1)(1 - \alpha)) \cdot (h(\mathbf{Y}_1^n | \mathbf{X}_2^n, W) - h(\mathbf{Y}_2^n | \mathbf{X}_2^n, W)),
\end{aligned}$$

and its perturbed version, for $\epsilon, \delta \in \mathbb{R}$,

$$\Theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n) := \sup_{\substack{p(v,w|\mathbf{x}_1^n,\mathbf{x}_2^n) \\ \mathbf{X}_1^n \to (V,W) \to \mathbf{X}_2^n \\ W \to V \to \mathbf{X}_2^n \\ \mathbf{X}_1^n \perp (V, \mathbf{X}_2^n)}} \theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W),$$

where

$$\theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W)$$

$$
\begin{aligned}
&:= (\lambda-1)\alpha \cdot I(\mathbf{X}_2^n; (\tilde{\mathbf{T}}_2^{(\delta)})^n|V) + (\lambda-1)(1-\alpha)\cdot \Big(h((\tilde{\mathbf{Y}}_2^{(\delta)})^n|W) - h(\mathbf{Y}_1^n|W)\Big) \\
&\quad + (1+(\lambda-1)(1-\alpha))\cdot \Big(h(\mathbf{Y}_1^n|\mathbf{X}_2^n, W) - h((\tilde{\mathbf{Y}}_2^{(\delta)})^n|\mathbf{X}_2^n, W)\Big) \\
&\quad - \epsilon \left(I(\mathbf{X}_2^n; (\tilde{\mathbf{T}}_2^{(\delta)})^n + \mathbf{G}^n|V) + h(\mathbf{Y}_1^n|W) + h(\mathbf{Y}_1^n|V,W)\right),
\end{aligned}
$$

where $\mathbf{G}_i$'s are independent Gaussian random variables of suitable dimension with identity covariance matrix, and the perturbed variables are defined by

$$
\begin{aligned}
\tilde{\mathbf{T}}_{2i}^{(\delta)} &:= \begin{pmatrix} 0 & I \\ 0 & \delta I \end{pmatrix} \begin{pmatrix} \mathbf{X}_{1i} \\ \mathbf{X}_{2i} \end{pmatrix} + \begin{pmatrix} \mathbf{Z}_{1i} \\ \hat{\mathbf{Z}}_{1i} \end{pmatrix} + \begin{pmatrix} \mathbf{Z}_{2i} \\ \hat{\mathbf{Z}}_{2i} \end{pmatrix}, \\
\tilde{\mathbf{Y}}_{2i}^{(\delta)} &:= \begin{pmatrix} I & I \\ 0 & \delta I \end{pmatrix} \begin{pmatrix} \mathbf{X}_{1i} \\ \mathbf{X}_{2i} \end{pmatrix} + \begin{pmatrix} \mathbf{Z}_{1i} \\ \hat{\mathbf{Z}}_{1i} \end{pmatrix} + \begin{pmatrix} \mathbf{Z}_{2i} \\ \hat{\mathbf{Z}}_{2i} \end{pmatrix},
\end{aligned}
$$

where $\hat{\mathbf{Z}}_{1i} \sim \mathcal{N}(0, N_1 I)$ and $\hat{\mathbf{Z}}_{2i} \sim \mathcal{N}(0, N_2 I)$ are Gaussian random variables in $\mathbb{R}^d$ mutually independent of each other and of $(\mathbf{Z}_1^n, \mathbf{Z}_2^n, \mathbf{X}_1^n, \mathbf{X}_2^n, V, W)$.

Note that for $\epsilon = \delta = 0$ the functionals $\theta_{\lambda,\alpha}^{(\epsilon,\delta)}$ and $\theta_{\lambda,\alpha}$ differ by only a constant

$$
\theta_{\lambda,\alpha}^{(0,0)}(\mathbf{X}_1^n, \mathbf{X}_2^n|V,W) = \theta_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n|V,W) - h(\hat{\mathbf{Z}}_1^n + \hat{\mathbf{Z}}_2^n).
$$

Thus $\theta_{\lambda,\alpha}^{(\epsilon,\delta)}$ can be viewed as a perburbation of $\theta_{\lambda,\alpha}$ via both ϵ and δ.

Proof of Theorem 4.2

Lemma 4.7. *Let $\epsilon, \delta \in \mathbb{R}$, $\lambda \geq 1$ and $0 \leq \alpha \leq 1$. Fix any distribution $p(\mathbf{x}_1^n, \mathbf{x}_2^n, v, w)$. Define the random variables*

$$
V_i := (V, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}), \tag{4.10a}
$$

$$
W_i := (W, \mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}), \tag{4.10b}
$$

for $i = 1, \ldots, n$. Then we have the followings:

(i) It holds that

$$
\begin{aligned}
&\theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n|V,W) \\
&= \sum_{i=1}^n \theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_{1i}, \mathbf{X}_{2i}|V_i, W_i) - \sum_{i=1}^n \Big((1+(\lambda-1)(1-\alpha)) I(\mathbf{Y}_{1i}; \mathbf{X}_2^{n\backslash i}|\mathbf{X}_{2i}, \tilde{\mathbf{Y}}_{2i}^{(\delta)}, W_i) \\
&\quad + \epsilon I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}; \tilde{\mathbf{T}}_{2i}^{(\delta)} + \mathbf{G}_i|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1} + \mathbf{G}^{i-1}, V) \\
&\quad + \epsilon I((\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W) + \epsilon I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, V, W) \Big).
\end{aligned}
$$

(ii) Suppose $\mathbf{X}_1^n \to (V,W) \to \mathbf{X}_2^n$ and $W \to V \to \mathbf{X}_2^n$ form Markov chains and $\mathbf{X}_1^n \perp (V, \mathbf{X}_2^n)$. Then $\mathbf{X}_{1i} \to (V_i, W_i) \to \mathbf{X}_{2i}$ and $W_i \to V_i \to \mathbf{X}_{2i}$ form Markov chains and $\mathbf{X}_{1i} \perp (V_i, \mathbf{X}_{2i})$ for $i = 1, \ldots, n$.

Proof of Lemma 4.7 (i). The proof follows from putting the following calculations together, with ($\star$) denoting application of chain rule and ($\sharp$) denoting application of Csiszár's sum identity.

(i) We have

$$\begin{aligned}
I(\mathbf{X}_2^n; (\tilde{\mathbf{T}}_2^{(\delta)})^n|V) &\overset{(\star)}{=} \sum_{i=1}^n I(\mathbf{X}_2^n; \tilde{\mathbf{T}}_{2i}^{(\delta)}|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) \\
&\overset{\text{(a)}}{=} \sum_{i=1}^n I(\mathbf{X}_{2i}; \tilde{\mathbf{T}}_{2i}^{(\delta)}|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) \\
&= \sum_{i=1}^n I(\mathbf{X}_{2i}; \tilde{\mathbf{T}}_{2i}^{(\delta)}|V_i),
\end{aligned}$$

where (a) holds since

$$\mathbf{X}_2^{n\backslash i} \to (\mathbf{X}_{2i}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) \to \tilde{\mathbf{T}}_{2i}^{(\delta)}$$

forms a Markov chain.

(ii) We have

$$\begin{aligned}
&h((\tilde{\mathbf{Y}}_2^{(\delta)})^n|W) - h(\mathbf{Y}_1^n|W) \\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|(\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) - h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W)\big) \\
&\overset{(\sharp)}{=} \sum_{i=1}^n \big(h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|(\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) - h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W) \\
&\qquad + I((\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W) - I(\mathbf{Y}_{1(i+1)}^n; \tilde{\mathbf{Y}}_{2i}^{(\delta)}|(\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W)\big) \\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) - h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W)\big) \\
&= \sum_{i=1}^n \big(h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|W_i) - h(\mathbf{Y}_{1i}|W_i)\big).
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&h(\mathbf{Y}_1^n|\mathbf{X}_2^n, W) - h((\tilde{\mathbf{Y}}_2^{(\delta)})^n|\mathbf{X}_2^n, W) \\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\mathbf{Y}_{1i}|\mathbf{X}_2^n, \mathbf{Y}_{1(i+1)}^n, W) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_2^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W)\big) \\
&\overset{(\sharp)}{=} \sum_{i=1}^n \big(h(\mathbf{Y}_{1i}|\mathbf{X}_2^n, \mathbf{Y}_{1(i+1)}^n, W) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_2^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) \\
&\qquad + I(\mathbf{Y}_{1(i+1)}^n; \tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_2^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) - I((\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{X}_2^n, \mathbf{Y}_{1(i+1)}^n, W)\big) \\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\mathbf{Y}_{1i}|\mathbf{X}_2^n, \mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_2^n, \mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \big(h(\mathbf{Y}_{1i}|\mathbf{X}_2^n, W_i) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_2^n, W_i) \big) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} \big(h(\mathbf{Y}_{1i}|\mathbf{X}_{2i}, W_i) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_{2i}, W_i) \\
&\qquad + I(\tilde{\mathbf{Y}}_{2i}^{(\delta)}; \mathbf{X}_2^{n\setminus i}|\mathbf{X}_{2i}, W_i) - I(\mathbf{Y}_{1i}; \mathbf{X}_2^{n\setminus i}|\mathbf{X}_{2i}, W_i) \big) \\
&\overset{\text{(a)}}{=} \sum_{i=1}^{n} \big(h(\mathbf{Y}_{1i}|\mathbf{X}_{2i}, W_i) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_{2i}, W_i) \\
&\qquad + I(\tilde{\mathbf{Y}}_{2i}^{(\delta)}; \mathbf{X}_2^{n\setminus i}|\mathbf{X}_{2i}, W_i) - I(\mathbf{Y}_{1i}, \tilde{\mathbf{Y}}_{2i}^{(\delta)}; \mathbf{X}_2^{n\setminus i}|\mathbf{X}_{2i}, W_i) \big) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} \big(h(\mathbf{Y}_{1i}|\mathbf{X}_{2i}, W_i) - h(\tilde{\mathbf{Y}}_{2i}^{(\delta)}|\mathbf{X}_{2i}, W_i) \big) - \sum_{i=1}^{n} I(\mathbf{Y}_{1i}; \mathbf{X}_2^{n\setminus i}|\mathbf{X}_{2i}, \tilde{\mathbf{Y}}_{2i}^{(\delta)}, W_i),
\end{aligned}
$$

where (a) holds since

$$
\tilde{\mathbf{Y}}_{2i}^{(\delta)} \to (\mathbf{Y}_{1i}, \mathbf{X}_{2i}, W_i) \to \mathbf{X}_2^{n\setminus i}
$$

forms a Markov chain.

(iv) Denoting $\mathbf{S}_{2i} := \tilde{\mathbf{T}}_{2i}^{(\delta)} + \mathbf{G}_i$ we have

$$
\begin{aligned}
&I(\mathbf{X}_2^n; (\tilde{\mathbf{T}}_2^{(\delta)})^n + \mathbf{G}^n|V) \\
&= I(\mathbf{X}_2^n; \mathbf{S}_2^n|V) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} I(\mathbf{X}_2^n; \mathbf{S}_{2i}|\mathbf{S}_2^{i-1}, V) \\
&\overset{\text{(a)}}{=} \sum_{i=1}^{n} I(\mathbf{X}_{2i}; \mathbf{S}_{2i}|\mathbf{S}_2^{i-1}, V) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} \big(I(\mathbf{X}_{2i}, \mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) - I(\mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) \big) \\
&\overset{\text{(b)}}{=} \sum_{i=1}^{n} \big(I(\mathbf{X}_{2i}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}; \mathbf{S}_{2i}|V) - I(\mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) \big) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} \big(I(\mathbf{X}_{2i}; \mathbf{S}_{2i}|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) + I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}; \mathbf{S}_{2i}|V) - I(\mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) \big) \\
&\overset{\text{(c)}}{=} \sum_{i=1}^{n} \big(I(\mathbf{X}_{2i}; \mathbf{S}_{2i}|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) + I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, \mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) - I(\mathbf{S}_2^{i-1}; \mathbf{S}_{2i}|V) \big) \\
&\overset{(\star)}{=} \sum_{i=1}^{n} \big(I(\mathbf{X}_{2i}; \mathbf{S}_{2i}|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) + I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}; \mathbf{S}_{2i}|\mathbf{S}_2^{i-1}, V) \big) \\
&= \sum_{i=1}^{n} I(\mathbf{X}_{2i}; \tilde{\mathbf{T}}_{2i}^{(\delta)} + \mathbf{G}_i|V_i) + \sum_{i=1}^{n} I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}; \tilde{\mathbf{T}}_{2i}^{(\delta)} + \mathbf{G}_i|(\tilde{\mathbf{T}}_2^{(\delta)})^{i-1} + \mathbf{G}^{i-1}, V),
\end{aligned}
$$

where (a) holds since

$$
\mathbf{X}_2^{n\setminus i} \to (\mathbf{X}_{2i}, \mathbf{S}_2^{i-1}, V) \to \mathbf{S}_{2i}
$$

forms a Markov chain, (b) holds since

$$((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, \mathbf{S}_2^{i-1}) \to (\mathbf{X}_{2i}, V) \to \mathbf{S}_{2i}$$

forms a Markov chain, (c) holds since

$$\mathbf{S}_2^{i-1} \to ((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V) \to \mathbf{S}_{2i}$$

forms a Markov chain.

(v) We have

$$\begin{aligned}
&h(\mathbf{Y}_1^n|W)\\
&\overset{(\star)}{=} \sum_{i=1}^n h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W)\\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, W) + I((\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W)\big)\\
&= \sum_{i=1}^n h(\mathbf{Y}_{1i}|W_i) + \sum_{i=1}^n I((\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, W).
\end{aligned}$$

(vi) We have

$$\begin{aligned}
&h(\mathbf{Y}_1^n|V, W)\\
&\overset{(\star)}{=} \sum_{i=1}^n h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, V, W)\\
&\overset{(\star)}{=} \sum_{i=1}^n \big(h(\mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, V, W)\\
&\qquad + I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, V, W)\big)\\
&= \sum_{i=1}^n h(\mathbf{Y}_{1i}|V_i, W_i) + \sum_{i=1}^n I((\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}; \mathbf{Y}_{1i}|\mathbf{Y}_{1(i+1)}^n, V, W).
\end{aligned}$$

□

Proof of Lemma 4.7 (ii). The Markov chain

$$\mathbf{X}_1^n \to (V, W) \to \mathbf{X}_2^n$$

implies

$$(\mathbf{X}_{1i}, \mathbf{Y}_{1(i+1)}^n, \mathbf{X}_1^{i-1}) \to (V, W) \to (\mathbf{X}_{2i}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1})$$

which in turn implies

$$\mathbf{X}_{1i} \to (V, W, \mathbf{Y}_{1(i+1)}^n, \mathbf{X}_1^{i-1}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}) \to \mathbf{X}_{2i}$$

which, since $\tilde{\mathbf{Y}}_{2i}^{(\delta)} = \tilde{\mathbf{T}}_{2i}^{(\delta)} + \begin{pmatrix} \mathbf{X}_{1i} \\ 0 \end{pmatrix}$, is the same as

$$\mathbf{X}_{1i} \to (V, W, \mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}) \to \mathbf{X}_{2i}$$

or equivalently

$$\mathbf{X}_{1i} \to (V_i, W_i) \to \mathbf{X}_{2i}.$$

Moreover, from both of the Markov chains in the assumption we have a Markov chain

$$(W, \mathbf{X}_1^n) \to V \to \mathbf{X}_2^n$$

which implies

$$(W, \mathbf{Y}_{1(i+1)}^n, \mathbf{X}_1^{i-1}) \to V \to (\mathbf{X}_{2i}, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1})$$

which in turn implies

$$(W, \mathbf{Y}_{1(i+1)}^n, \mathbf{X}_1^{i-1}) \to (V, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}) \to \mathbf{X}_{2i}$$

which, since $\tilde{\mathbf{Y}}_{2i}^{(\delta)} = \tilde{\mathbf{T}}_{2i}^{(\delta)} + \begin{pmatrix} \mathbf{X}_{1i} \\ 0 \end{pmatrix}$, is the same as

$$(W, \mathbf{Y}_{1(i+1)}^n, (\tilde{\mathbf{Y}}_2^{(\delta)})^{i-1}) \to (V, (\tilde{\mathbf{T}}_2^{(\delta)})^{i-1}) \to \mathbf{X}_{2i}$$

or equivalently

$$W_i \to V_i \to \mathbf{X}_{2i}.$$

Finally, $\mathbf{X}_{1i} \perp (V_i, \mathbf{X}_{2i})$ is immediate from $\mathbf{X}_1^n \perp (V, \mathbf{X}_2^n)$. □

Lemma 4.8. *Let $\lambda \geq 1$ and $0 \leq \alpha \leq 1$. Let $n \geq 1$. Let $\mathcal{D}$ be any set of distributions $p(\mathbf{x}_1^n)p(\mathbf{x}_2^n)$ such that $\mathrm{E}[\|\mathbf{X}_1^n\|^2]$ and $\mathrm{E}[\|\mathbf{X}_2^n\|^2]$ are bounded. Then*

$$\lim_{(\epsilon,\delta)\to(0,0)} \sup_{p(\mathbf{x}_1^n)p(\mathbf{x}_2^n)\in\mathcal{D}} \Theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n) = \sup_{p(\mathbf{x}_1^n)p(\mathbf{x}_2^n)\in\mathcal{D}} \Theta_{\lambda,\alpha}^{(0,0)}(\mathbf{X}_1^n, \mathbf{X}_2^n).$$

Proof. By applying Lemma 4.9 one can show, after some computations, that

$$\sup_{\substack{p(\mathbf{x}_1^n, \mathbf{x}_2^n, v, w) \\ p(\mathbf{x}_1^n, \mathbf{x}_2^n)\in\mathcal{D} \\ \mathbf{X}_1^n \to (V,W) \to \mathbf{X}_2^n \\ W \to V \to \mathbf{X}_2^n \\ \mathbf{X}_1^n \perp (V, \mathbf{X}_2^n)}} \left| \theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W) - \theta_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W) \right| = O(|\epsilon| + \delta^2).$$

The result then follows from Lemma 4.10. □

Proposition 4.5. *Let $\lambda \geq 1$ and $0 \leq \alpha \leq 1$. Let $\mathcal{K}_{1i}, \mathcal{K}_{2i}$ $(i = 1, \ldots, n)$ be sets of $d \times d$ matrices $(d \geq 1)$. Denote*

$$v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\ldots,(\mathcal{K}_{1n},\mathcal{K}_{2n})} := \sup_{\substack{p(\mathbf{x}_1^n)p(\mathbf{x}_2^n) \\ \mathrm{E}[\mathbf{X}_1^n]=\mathrm{E}[\mathbf{X}_2^n]=0 \\ \mathrm{Cov}(\mathbf{X}_{1i})\in\mathcal{K}_{1i},\, \mathrm{Cov}(\mathbf{X}_{2i})\in\mathcal{K}_{2i}}} \Theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n).$$

Then we have the followings:

(i) Let $\epsilon \geq 0$ and $\delta \in \mathbb{R}$. Then

$$\sum_{i=1}^n v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})} = v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\ldots,(\mathcal{K}_{1n},\mathcal{K}_{2n})}.$$

(ii) Let $\epsilon \geq 0$ and $\delta \in \mathbb{R}$. For any $0 \leq t \leq 1$ we have

$$v^{(1)}_{(\mathcal{K}_{11},\mathcal{K}_{21})} + v^{(1)}_{(\mathcal{K}_{12},\mathcal{K}_{22})} \leq v^{(1)}_{(\mathcal{K}^t_{11},\mathcal{K}^t_{21})} + v^{(1)}_{(\mathcal{K}^t_{12},\mathcal{K}^t_{22})},$$

where the sets of $d \times d$ matrices $\mathcal{K}^t_{11}, \mathcal{K}^t_{21}, \mathcal{K}^t_{12}, \mathcal{K}^t_{22}$ are defined by

$$\mathcal{K}^t_{11} := t\mathcal{K}_{11} + (1-t)\mathcal{K}_{12}, \qquad \mathcal{K}^t_{21} := t\mathcal{K}_{21} + (1-t)\mathcal{K}_{22},$$
$$\mathcal{K}^t_{12} := (1-t)\mathcal{K}_{11} + t\mathcal{K}_{12}, \qquad \mathcal{K}^t_{22} := (1-t)\mathcal{K}_{21} + t\mathcal{K}_{22}.$$

(iii) Let $\epsilon > 0$ and $\delta \neq 0$, or let $\epsilon = \delta = 0$. Suppose $\mathcal{K}_1, \mathcal{K}_2$ are compact convex sets of $d \times d$ matrices. Then there exists a maximizer $p^*(\mathbf{x}_1, \mathbf{x}_2, v, w)$ for $v^{(1)}_{(\mathcal{K}_1,\mathcal{K}_2)}$ such that $p^*(\mathbf{x}_1|w)$, $p^*(\mathbf{x}_1|v, w)$ and $p^*(\mathbf{x}_2|v)$ are Gaussians with covariance matrices independent of choice of v and w.

Proof of Proposition 4.5 (i). We shall first show

$$\sum_{i=1}^n v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})} \leq v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\ldots,(\mathcal{K}_{1n},\mathcal{K}_{2n})}.$$

Suppose $(\mathbf{X}^*_{1i}, \mathbf{X}^*_{2i}, V^*_i, W^*_i)$ are random variables satisfying the constraints of $v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})}$, and are mutually independent among $i = 1, \ldots, n$. Then the random variables defined by

$$(\mathbf{X}_1^n, \mathbf{X}_2^n, V, W) := ((\mathbf{X}^*_1)^n, (\mathbf{X}^*_2)^n, (V^*)^n, (W^*)^n)$$

satisfy the constraints of $v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\ldots,(\mathcal{K}_{1n},\mathcal{K}_{2n})}$, as well as

$$\sum_{i=1}^n \theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}^*_{1i}, \mathbf{X}^*_{2i}|V^*_i, W^*_i) \overset{\text{(a)}}{=} \theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n|V, W) \leq v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\ldots,(\mathcal{K}_{1n},\mathcal{K}_{2n})},$$

where (a) follows from the additivity of entropy for independent random variables.

Now we show

$$\sum_{i=1}^{n} v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})} \geq v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\dots,(\mathcal{K}_{1n},\mathcal{K}_{2n})}.$$

Suppose $(\mathbf{X}_1^n, \mathbf{X}_2^n, V, W)$ are random variables satisfying the constraints of $v^{(n)}_{(\mathcal{K}_{11},\mathcal{K}_{21}),\dots,(\mathcal{K}_{1n},\mathcal{K}_{2n})}$. Then

$$\theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_1^n, \mathbf{X}_2^n | V, W) \overset{\text{(a)}}{\leq} \sum_{i=1}^{n} \theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_{1i}, \mathbf{X}_{2i} | V_i, W_i) \overset{\text{(b)}}{\leq} \sum_{i=1}^{n} v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})},$$

where V_i, W_i are defined by (4.10), (a) follows from Lemma 4.7 (i) and we have used $\epsilon \geq 0$, and (b) holds since $(\mathbf{X}_{1i}, \mathbf{X}_{2i}, V_i, W_i)$ satisfies the constraints of $v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})}$, as a result of Lemma 4.7 (ii). □

Proof of Proposition 4.5 (ii). Suppose $(\mathbf{X}^*_{1i}, \mathbf{X}^*_{2i}, V^*_i, W^*_i)$ are random variables satisfying the constraints of $v^{(1)}_{(\mathcal{K}_{1i},\mathcal{K}_{2i})}$, and are independent among $i = 1, 2$. Let

$$(V, W) := ((V_1^*, V_2^*), (W_1^*, W_2^*)), \tag{4.11a}$$

$$\begin{pmatrix} \mathbf{X}_{11} \\ \mathbf{X}_{12} \end{pmatrix} := \begin{pmatrix} \sqrt{t}I & \sqrt{1-t}I \\ -\sqrt{1-t}I & \sqrt{t}I \end{pmatrix} \begin{pmatrix} \mathbf{X}^*_{11} \\ \mathbf{X}^*_{12} \end{pmatrix}, \tag{4.11b}$$

$$\begin{pmatrix} \mathbf{X}_{21} \\ \mathbf{X}_{22} \end{pmatrix} := \begin{pmatrix} \sqrt{t}I & \sqrt{1-t}I \\ -\sqrt{1-t}I & \sqrt{t}I \end{pmatrix} \begin{pmatrix} \mathbf{X}^*_{21} \\ \mathbf{X}^*_{22} \end{pmatrix}. \tag{4.11c}$$

It is immediate that the distribution of $((\mathbf{X}_{11}, \mathbf{X}_{12}), (\mathbf{X}_{21}, \mathbf{X}_{22}), V, W)$ satisfies the assumption in Lemma 4.7 (ii) and hence the distribution of $(\mathbf{X}_{1i}, \mathbf{X}_{2i}, V_i, W_i)$, where V_i, W_i are defined by (4.10), satisfies the constraints of $v^{(1)}_{(\mathcal{K}^t_{1i},\mathcal{K}^t_{2i})}$. Now we have

$$\begin{aligned}
&\theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}^*_{11}, \mathbf{X}^*_{21} | V_1^*, W_1^*) + \theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}^*_{12}, \mathbf{X}^*_{22} | V_2^*, W_2^*) \\
&\overset{\text{(a)}}{=} \theta^{(\epsilon,\delta)}_{\lambda,\alpha}((\mathbf{X}^*_{11}, \mathbf{X}^*_{12}), (\mathbf{X}^*_{21}, \mathbf{X}^*_{22}) | V, W) \\
&\overset{\text{(b)}}{=} \theta^{(\epsilon,\delta)}_{\lambda,\alpha}((\mathbf{X}_{11}, \mathbf{X}_{12}), (\mathbf{X}_{21}, \mathbf{X}_{22}) | V, W) \\
&\overset{\text{(c)}}{=} \underbrace{\theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_{11}, \mathbf{X}_{21} | V_1, W_1)}_{\leq v^{(1)}_{(\mathcal{K}^t_{11},\mathcal{K}^t_{21})}} + \underbrace{\theta^{(\epsilon,\delta)}_{\lambda,\alpha}(\mathbf{X}_{12}, \mathbf{X}_{22} | V_2, W_2)}_{\leq v^{(1)}_{(\mathcal{K}^t_{12},\mathcal{K}^t_{22})}} \\
&\qquad - (1 + (\lambda - 1)(1 - \alpha)) \cdot I(\mathbf{Y}_{11}; \mathbf{X}_{22} | \mathbf{X}_{21}, \tilde{\mathbf{Y}}^{(\delta)}_{21}, W_1) \\
&\qquad - (1 + (\lambda - 1)(1 - \alpha)) \cdot I(\mathbf{Y}_{12}; \mathbf{X}_{21} | \mathbf{X}_{22}, \tilde{\mathbf{Y}}^{(\delta)}_{22}, W_2) \\
&\qquad - \epsilon I(\tilde{\mathbf{T}}^{(\delta)}_{21}; \tilde{\mathbf{T}}^{(\delta)}_{22} + \mathbf{G}_2 | \tilde{\mathbf{T}}^{(\delta)}_{21} + \mathbf{G}_1, V) - \epsilon I(\tilde{\mathbf{Y}}^{(\delta)}_{21}; \mathbf{Y}_{12} | W) \\
&\qquad - \epsilon I(\tilde{\mathbf{T}}^{(\delta)}_{21}, \tilde{\mathbf{Y}}^{(\delta)}_{21}; \mathbf{Y}_{12} | V, W) \\
&\leq v^{(1)}_{(\mathcal{K}^t_{11},\mathcal{K}^t_{21})} + v^{(1)}_{(\mathcal{K}^t_{12},\mathcal{K}^t_{22})},
\end{aligned} \tag{4.12}$$

where (a) follows from the additivity of entropy for independent random variables, (b) follows from rotational invariance of entropy, and (c) is established by Lemma 4.7 (i). □

Proof of Proposition 4.5 (iii). The existence of maximizer can be justified using Lemma 4.4, 4.5 and 4.6 through standard techniques in Appendix II of [GN14].

Now consider $\epsilon > 0$ and $\delta \neq 0$. The proof follows the same lines of reasoning in the proof of Proposition 4.5 (ii), with the choice $\mathcal{K}_{1i} = \mathcal{K}_1$ and $\mathcal{K}_{2i} = \mathcal{K}_2$, from which we have $\mathcal{K}_{1i}^t = \mathcal{K}_1$ and $\mathcal{K}_{2i}^t = \mathcal{K}_2$ for any choice of t, since $\mathcal{K}_1, \mathcal{K}_2$ are convex sets. Take two independent copies $(\mathbf{X}_{11}^*, \mathbf{X}_{21}^*, V_1^*, W_1^*)$ and $(\mathbf{X}_{12}^*, \mathbf{X}_{22}^*, V_2^*, W_2^*)$ of a maximizer $p^*(\mathbf{x}_1, \mathbf{x}_2, v, w)$ for $v_{(\mathcal{K}_1,\mathcal{K}_2)}^{(1)}$. With $t = \frac{1}{2}$ define $((\mathbf{X}_{11}, \mathbf{X}_{12}), (\mathbf{X}_{21}, \mathbf{X}_{22}), V, W)$ as in (4.11). Following the steps in (4.12) we have

$$\begin{aligned}
2v_{(\mathcal{K}_1,\mathcal{K}_2)}^{(1)} &= \theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_{11}^*, \mathbf{X}_{21}^* | V_1^*, W_1^*) + \theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_{12}^*, \mathbf{X}_{22}^* | V_2^*, W_2^*) \\
&= \underbrace{\theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_{11}, \mathbf{X}_{21} | V_1, W_1)}_{\leq v_{(\mathcal{K}_1,\mathcal{K}_2)}^{(1)}} + \underbrace{\theta_{\lambda,\alpha}^{(\epsilon,\delta)}(\mathbf{X}_{12}, \mathbf{X}_{22} | V_2, W_2)}_{\leq v_{(\mathcal{K}_1,\mathcal{K}_2)}^{(1)}} \\
&\quad - (1 + (\lambda - 1)(1 - \alpha)) \cdot I(\mathbf{Y}_{11}; \mathbf{X}_{22} | \mathbf{X}_{21}, \tilde{\mathbf{Y}}_{21}^{(\delta)}, W_1) \\
&\quad - (1 + (\lambda - 1)(1 - \alpha)) \cdot I(\mathbf{Y}_{12}; \mathbf{X}_{21} | \mathbf{X}_{22}, \tilde{\mathbf{Y}}_{22}^{(\delta)}, W_2) \\
&\quad - \epsilon I(\tilde{\mathbf{T}}_{21}^{(\delta)}; \tilde{\mathbf{T}}_{22}^{(\delta)} + \mathbf{G}_2 | \tilde{\mathbf{T}}_{21}^{(\delta)} + \mathbf{G}_1, V) - \epsilon I(\tilde{\mathbf{Y}}_{21}^{(\delta)}; \mathbf{Y}_{12} | W) \\
&\quad - \epsilon I(\tilde{\mathbf{T}}_{21}^{(\delta)}, \tilde{\mathbf{Y}}_{21}^{(\delta)}; \mathbf{Y}_{12} | V, W) \\
&\leq 2v_{(\mathcal{K}_1,\mathcal{K}_2)}^{(1)}.
\end{aligned}$$

Non-negativity of mutual information forces the Markov chains

$$\tilde{\mathbf{T}}_{21}^{(\delta)} \to (\tilde{\mathbf{T}}_{21}^{(\delta)} + \mathbf{G}_1, V) \to \tilde{\mathbf{T}}_{22}^{(\delta)} + \mathbf{G}_2, \tag{4.13}$$

$$\tilde{\mathbf{Y}}_{21}^{(\delta)} \to W \to \mathbf{Y}_{12}, \tag{4.14}$$

$$(\tilde{\mathbf{T}}_{21}^{(\delta)}, \tilde{\mathbf{Y}}_{21}^{(\delta)}) \to (V, W) \to \mathbf{Y}_{12}, \tag{4.15}$$

where (4.13) together with the Markov chain

$$\tilde{\mathbf{T}}_{21}^{(\delta)} + \mathbf{G}_1 \to (\tilde{\mathbf{T}}_{21}^{(\delta)}, V) \to \tilde{\mathbf{T}}_{22}^{(\delta)} + \mathbf{G}_2$$

implies by double Markovity (Lemma 4.1) that

$$(\tilde{\mathbf{T}}_{21}^{(\delta)}, \tilde{\mathbf{T}}_{21}^{(\delta)} + \mathbf{G}_1) \to V \to \tilde{\mathbf{T}}_{22}^{(\delta)} + \mathbf{G}_2 \tag{4.16}$$

forms a Markov chain. Using the fact that $\delta \neq 0$, applying Lemma 4.2 to (4.16), (4.14), (4.15), respectively, gives the Markov chains

$$\begin{aligned}
\mathbf{X}_{21} &\to V \to \mathbf{X}_{22}, \\
\mathbf{X}_{11} &\to W \to \mathbf{X}_{12}, \\
\mathbf{X}_{11} &\to (V, W) \to \mathbf{X}_{12},
\end{aligned}$$

which, by Lemma 4.3, imply that for any $v_1^*, v_2^*, w_1^*, w_2^*$ each of the pairs of conditional distributions

$$(\mathbf{X}_{21}^*|_{V_1^*=v_1^*}, \mathbf{X}_{22}^*|_{V_2^*=v_2^*}),$$

$$(\mathbf{X}_{11}^*|_{W_1^*=w_1^*}, \mathbf{X}_{12}^*|_{W_2^*=w_2^*}),$$
$$(\mathbf{X}_{11}^*|_{V_1^*=v_1^*, W_1^*=w_1^*}, \mathbf{X}_{12}^*|_{V_2^*=v_2^*, W_2^*=w_2^*})$$

consists of Gaussians with the same covariance matrix. Since $v_1^*, v_2^*, w_1^*, w_2^*$ are arbitrary, we can conclude that $p^*(\mathbf{x}_1|w)$, $p^*(\mathbf{x}_1|v,w)$ and $p^*(\mathbf{x}_2|v)$ are Gaussians with covariance matrices independent of choice of v and w.

Finally we show the case $\epsilon = \delta = 0$. Let $\tilde{v}^{(1)}_{(\mathcal{K}_1,\mathcal{K}_2)}$ be defined in the same way as $v^{(1)}_{(\mathcal{K}_1,\mathcal{K}_2)}$ but with the additional constraint that $p(\mathbf{x}_1^n|w)$, $p(\mathbf{x}_1^n|v,w)$ and $p(\mathbf{x}_2^n|v)$ are Gaussians with covariance matrices independent of choice of v and w. Then we have $\tilde{v}^{(1)}_{(\mathcal{K}_1,\mathcal{K}_2)} = v^{(1)}_{(\mathcal{K}_1,\mathcal{K}_2)}$ for $\epsilon > 0$ and $\delta \neq 0$. One can take the limit $(\epsilon, \delta) \to (0,0)$ and apply Lemma 4.8 to get the result. □

Proposition 4.6. *Let $\lambda \geq 1$ and $0 \leq \alpha \leq 1$. Let $\mathcal{K}_1, \mathcal{K}_2$ be compact convex sets of $d \times d$ matrices ($d \geq 1$). Then*

$$\sup_{\substack{p(\mathbf{x}_1)p(\mathbf{x}_2) \\ \mathrm{E}[\mathbf{X}_1]=\mathrm{E}[\mathbf{X}_2]=0 \\ \mathrm{Cov}(\mathbf{X}_1)\in\mathcal{K}_1,\, \mathrm{Cov}(\mathbf{X}_2)\in\mathcal{K}_2}} \Theta_{\lambda,\alpha}(\mathbf{X}_1,\mathbf{X}_2) = \sup_{\substack{A_2,B_1,B_2,C_1,C_2,\Sigma\in\mathbb{R}^{d\times d} \\ A_2, \begin{pmatrix} B_1 & \Sigma \\ \Sigma^T & B_2 \end{pmatrix}, \begin{pmatrix} C_1 & -\Sigma \\ -\Sigma^T & C_2 \end{pmatrix} \succeq 0 \\ B_1+C_1\in\mathcal{K}_1,\, A_2+B_2+C_2\in\mathcal{K}_2}} g_{\lambda,\alpha}(A_2,B_1,B_2,\Sigma), \tag{4.17}$$

where

$$\begin{aligned}
&g_{\lambda,\alpha}(A_2,B_1,B_2,\Sigma) \\
&:= \frac{1}{2}\Big((\lambda-1)\alpha\left(\log|A_2+N_1I+N_2I| - \log|N_1I+N_2I|\right) \\
&\quad + (\lambda-1)(1-\alpha)\big(\log|B_1+B_2+\Sigma+\Sigma^T+A_2+N_1I+N_2I| \\
&\qquad - \log|B_1+N_1I|\big) \\
&\quad + (1+(\lambda-1)(1-\alpha))\big(\log|B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T+N_1I| \\
&\qquad - \log|B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T+N_1I+N_2I|\big)\Big).
\end{aligned} \tag{4.18}$$

Proof. Denote the left-hand side and right-hand side of (4.17) by v_L and v_R, respectively.

We first show $v_L \leq v_R$. By Proposition 4.5 (iii) v_L admits a maximizing distribution $(\mathbf{X}_1, \mathbf{X}_2, V, W)$ such that $p(\mathbf{x}_1|w)$, $p(\mathbf{x}_1|v,w)$ and $p(\mathbf{x}_2|v)$ are Gaussians with covariance matrices independent of choice of v and w. Let

$$\begin{aligned}
A_2 &:= \mathrm{E}[\mathrm{Cov}(\mathbf{X}_2|V)], \\
\begin{pmatrix} B_1 & \Sigma \\ \Sigma^T & B_2 \end{pmatrix} &:= \mathrm{E}\left[\mathrm{Cov}\left(\left(\begin{pmatrix}\mathbf{X}_1 \\ \mathbf{X}_2\end{pmatrix}\middle| W\right)\right)\right] - \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \\
C_1 &:= \mathrm{Cov}(\mathbf{X}_1) - B_1, \\
C_2 &:= \mathrm{Cov}(\mathbf{X}_2) - B_2 - A_2.
\end{aligned}$$

One can then verify

$$\begin{aligned}
&h(\mathbf{X}_2+\mathbf{Z}_1+\mathbf{Z}_2|V)-h(\mathbf{Z}_1+\mathbf{Z}_2)\\
&\overset{\text{(a)}}{\leq}\frac{1}{2}\left(\log|A_2+N_1I+N_2I|-\log|N_1I+N_2I|\right),\\
&h(\mathbf{X}_2+\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2|W)-h(\mathbf{X}_1+\mathbf{Z}_1|W)\\
&\overset{\text{(b)}}{\leq}\frac{1}{2}\big(\log|B_1+B_2+\Sigma+\Sigma^T+A_2+N_1I+N_2I|-\log|B_1+N_1I|\big),\\
&h(\mathbf{X}_1+\mathbf{Z}_1|\mathbf{X}_2,W)-h(\mathbf{X}_1+\mathbf{Z}_1+\mathbf{Z}_2|\mathbf{X}_2,W)\\
&\overset{\text{(c)}}{\leq}\frac{1}{2}\left(\log|\operatorname{E}[\operatorname{Cov}(\mathbf{X}_1|\mathbf{X}_2,W)]+N_1I|-\log|\operatorname{E}[\operatorname{Cov}(\mathbf{X}_1|\mathbf{X}_2,W)]+N_1I+N_2I|\right)\\
&\overset{\text{(d)}}{\leq}\frac{1}{2}\big(\log|B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T+N_1I|\\
&\qquad-\log|B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T+N_1I+N_2I|\big),
\end{aligned}$$

where (a) holds since $\mathbf{X}_2|_{V=v}$ is Gaussian of covariance A_2 for all v, (b) follows from Lemma 4.13 and that $\mathbf{X}_1|_{W=w}$ is Gaussian of covariance B_1 for all w, (c) follows from Lemma 4.12, and (d) follows from Lemma 4.11 and that

$$\operatorname{E}[\operatorname{Cov}(\mathbf{X}_1|\mathbf{X}_2,W)]\preceq B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T,$$

which is shown below. Observe that the orthogonality property of conditional expectation (implying also the vector extension of minimum mean square error property) yields the following:

$$\begin{aligned}
\operatorname{E}[\operatorname{Cov}(\mathbf{X}_1|\mathbf{X}_2,W)]&=\operatorname{E}[(\mathbf{X}_1-\operatorname{E}[\mathbf{X}_1|\mathbf{X}_2,W])(\mathbf{X}_1-\operatorname{E}[\mathbf{X}_1|\mathbf{X}_2,W])^T]\\
&\preceq\operatorname{E}[(\mathbf{X}_1-\tilde{\mathbf{X}}_1)(\mathbf{X}_1-\tilde{\mathbf{X}}_1)^T]
\end{aligned}$$

for any $\tilde{\mathbf{X}}_1$ that is $\sigma(\mathbf{X}_2,W)$ measurable and $\operatorname{E}[\|\tilde{\mathbf{X}}_1\|^2]<\infty$. In particular we set

$$\tilde{\mathbf{X}}_1:=\mathbf{X}_1-\hat{\mathbf{X}}_1+\operatorname{E}[\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2^T]\operatorname{E}[\hat{\mathbf{X}}_2\hat{\mathbf{X}}_2^T]^{-1}\hat{\mathbf{X}}_2,$$

where $\hat{\mathbf{X}}_1:=\mathbf{X}_1-\operatorname{E}[\mathbf{X}_1|W]$ and $\hat{\mathbf{X}}_2:=\mathbf{X}_2-\operatorname{E}[\mathbf{X}_2|W]$. Then

$$\begin{aligned}
\operatorname{E}[(\mathbf{X}_1-\tilde{\mathbf{X}}_1)(\mathbf{X}_1-\tilde{\mathbf{X}}_1)^T]&=\operatorname{E}[\hat{\mathbf{X}}_1\hat{\mathbf{X}}_1^T]-\operatorname{E}[\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2^T]\operatorname{E}[\hat{\mathbf{X}}_2\hat{\mathbf{X}}_2^T]^{-1}\operatorname{E}[\hat{\mathbf{X}}_2\hat{\mathbf{X}}_1^T]\\
&=B_1-\Sigma(A_2+B_2)^{-1}\Sigma^T.
\end{aligned}$$

Putting these together gives

$$\theta_{\lambda,\alpha}(\mathbf{X}_1,\mathbf{X}_2|V,W)\leq g_{\lambda,\alpha}(A_2,B_1,B_2,\Sigma).$$

Now we verify $A_2,B_1,B_2,C_1,C_2,\Sigma$ satisfy the constraints of v_R. We have

$$\operatorname{E}\left[\operatorname{Cov}\left(\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\end{pmatrix}\middle|W\right)\right]\overset{\text{(a)}}{\succeq}\operatorname{E}\left[\operatorname{Cov}\left(\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\end{pmatrix}\middle|V,W\right)\right]$$

$$
\begin{aligned}
&\overset{\text{(b)}}{=} \begin{pmatrix} \mathrm{E}[\mathrm{Cov}(\mathbf{X}_1|V,W)] & 0 \\ 0 & \mathrm{E}[\mathrm{Cov}(\mathbf{X}_2|V)] \end{pmatrix} \\
&\succeq \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix},
\end{aligned}
$$

where (a) follows from Lemma 4.14 and (b) holds since $\mathbf{X}_1 \to (V, W) \to \mathbf{X}_2$ and $W \to V \to \mathbf{X}_2$ form Markov chains. This gives

$$
\begin{pmatrix} B_1 & \Sigma \\ \Sigma^T & B_2 \end{pmatrix} \succeq 0.
$$

We also have

$$
\begin{aligned}
\begin{pmatrix} C_1 & -\Sigma \\ -\Sigma^T & C_2 \end{pmatrix} &= \begin{pmatrix} \mathrm{Cov}(\mathbf{X}_1) & 0 \\ 0 & \mathrm{Cov}(\mathbf{X}_2) \end{pmatrix} - \mathrm{E}\left[\mathrm{Cov}\left(\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \middle| W\right)\right] \\
&\overset{\text{(a)}}{=} \mathrm{Cov}\left(\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\right) - \mathrm{E}\left[\mathrm{Cov}\left(\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \middle| W\right)\right] \\
&\overset{\text{(b)}}{\succeq} 0,
\end{aligned}
$$

where (a) holds since $\mathbf{X}_1 \perp \mathbf{X}_2$ and (b) follows from Lemma 4.14. The remaining constraints, namely,

$$
\begin{aligned}
A_2 &\succeq 0, \\
B_1 + C_1 &\in \mathcal{K}_1, \\
A_2 + B_2 + C_2 &\in \mathcal{K}_2,
\end{aligned}
$$

are obvious.

Next we show $v_L \geq v_R$. Suppose $A_2, B_1, B_2, C_1, C_2, \Sigma$ are matrices that satisfy the constraints of v_R. Let

$$
\begin{pmatrix} \mathbf{A}_2 \\ \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} A_2 & & & & \\ & B_1 & \Sigma & & \\ & \Sigma^T & B_2 & & \\ & & & C_1 & -\Sigma \\ & & & -\Sigma^T & C_2 \end{pmatrix}\right),
$$

and let

$$
\begin{aligned}
\mathbf{X}_1 &:= \mathbf{B}_1 + \mathbf{C}_1, \\
\mathbf{X}_2 &:= \mathbf{A}_2 + \mathbf{B}_2 + \mathbf{C}_2, \\
V &:= \mathbf{B}_2 + \mathbf{C}_2, \\
W &:= (\mathbf{C}_1, \mathbf{C}_2).
\end{aligned}
$$

Then one can readily verify

$$\theta_{\lambda,\alpha}(\mathbf{X}_1, \mathbf{X}_2|V, W) = g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma).$$

and that $(\mathbf{X}_1, \mathbf{X}_2, V, W)$ satisfies the constraints of v_L. $\square$

Proposition 4.7. *Let $\lambda \geq 1$ and $0 \leq \alpha \leq 1$. Define the functional*

$$G^{(n)}_{\lambda,\alpha}(P_1, P_2) := \sup_{\substack{p(x_1^n)p(x_2^n): \\ \mathrm{E}[X_1^n]=\mathrm{E}[X_2^n]=0 \\ \mathrm{E}[\|X_1^n\|^2]\leq nP_1,\, \mathrm{E}[\|X_2^n\|^2]\leq nP_2}} \Theta_{\lambda,\alpha}(X_1^n, X_2^n),$$

for $n \geq 1$ and $P_1, P_2 \geq 0$, where X_{1i}, X_{2i} $(i = 1, \ldots, n)$ are random variables in $\mathbb{R}$. Then we have the followings:

(i) $G^{(n)}_{\lambda,\alpha}(P_1, P_2) = n \cdot G^{(1)}_{\lambda,\alpha}(P_1, P_2)$.

(ii) For $P_1, P_2 > 0$,

$$G^{(n)}_{\lambda,\alpha}(P_1, P_2) = n \cdot \sup_{\substack{B_1, B_2, \Sigma \geq 0 \\ \Sigma \leq \sqrt{B_1 B_2} \\ \frac{B_1}{P_1} + \frac{B_2}{P_2} \leq 1}} g_{\lambda,\alpha}(P_2 - \frac{P_1 B_2}{P_1 - B_1}, B_1, B_2, \Sigma),$$

where $g_{\lambda,\alpha}$ is defined as in (4.18), and $\frac{0}{0}$ is understood to be 0.

(iii) If $\lambda \geq \lambda_0$, where

$$\lambda_0 := 1 + \frac{N_2}{N_1} \frac{1}{(1 - \sqrt{\frac{N_1+N_2}{P_2+N_1+N_2}})^2},$$

then there exists $0 \leq \alpha \leq 1$ such that the maximization on right-hand side of (ii) is attained by $B_1 = B_2 = \Sigma = 0$.

(iv) If $\lambda \geq \lambda_0$ where λ_0 is defined as in (iii), then

$$\inf_{0 \leq \alpha \leq 1} G^{(n)}_{\lambda,\alpha}(P_1, P_2) \leq \frac{n}{2} \left((\lambda - 1) \log(P_2 + N_1 + N_2) + \log N_1 - \lambda \log(N_1 + N_2) \right).$$

Proof of Proposition 4.7 (i). Proposition 4.5 (i) (with $\mathcal{K}_{ji} := \{K : \mathrm{tr}(K) \leq P_j\}$) implies that

$$G^{(n)}_{\lambda,\alpha}(P_1, P_2) \geq n \cdot G^{(1)}_{\lambda,\alpha}(P_1, P_2),$$

as well as (with $\mathcal{K}_{ji} := \{K : \mathrm{tr}(K) \leq Q_{ji}\}$) that

$$G^{(n)}_{\lambda,\alpha}(P_1, P_2) \leq \sup_{\substack{Q_{1i}, Q_{2i} \geq 0 \\ \sum_{i=1}^n Q_{1i} \leq nP_1 \\ \sum_{i=1}^n Q_{2i} \leq nP_2}} \sum_{i=1}^n G^{(1)}_{\lambda,\alpha}(Q_{1i}, Q_{2i}).$$

It then suffices to show that $(P_1, P_2) \mapsto G^{(1)}_{\lambda,\alpha}(P_1, P_2)$ is concave. Indeed Proposition 4.5 (ii) (with $\mathcal{K}_{ji} := \{K : \mathrm{tr}(K) \le P_{ji}\}$ and $t := \frac{1}{2}$) implies

$$G^{(1)}_{\lambda,\alpha}(P_{11}, P_{21}) + G^{(1)}_{\lambda,\alpha}(P_{12}, P_{22}) \le 2 \cdot G^{(1)}_{\lambda,\alpha}\left(\frac{P_{11} + P_{12}}{2}, \frac{P_{21} + P_{22}}{2}\right)$$

for any $P_{1i}, P_{2i} \ge 0$ $(i = 1, 2)$, i.e., $G^{(1)}_{\lambda,\alpha}$ is midpoint-concave. This together with the fact that $G^{(1)}_{\lambda,\alpha}$ is continuous, which can be shown by Lemma 4.15 by considering the matrix expression (4.17), implies that $G^{(1)}_{\lambda,\alpha}$ is concave. □

Proof of Proposition 4.7 (ii). In view of (i) it suffices to show the scalar case, i.e., $n = 1$. Proposition 4.6 gives

$$G^{(1)}_{\lambda,\alpha}(P_1, P_2) = \sup_{\substack{A_2, B_1, B_2, C_1, C_2 \ge 0,\, \Sigma \in \mathbb{R} \\ \Sigma^2 \le B_1 B_2 \\ \Sigma^2 \le C_1 C_2 \\ B_1 + C_1 \le P_1 \\ A_2 + B_2 + C_2 \le P_2}} g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma),$$

where $g_{\lambda,\alpha}$ is defined as in (4.18) for scalars:

$$\begin{aligned} g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma) = \frac{1}{2}\Bigg(&(\lambda - 1)\alpha \log \frac{A_2 + N_1 + N_2}{N_1 + N_2} \\ &+ (\lambda - 1)(1 - \alpha) \log \frac{B_1 + B_2 + 2\Sigma + A_2 + N_1 + N_2}{B_1 + N_1} \\ &+ (1 + (\lambda - 1)(1 - \alpha)) \log \frac{B_1 - \frac{\Sigma^2}{A_2 + B_2} + N_1}{B_1 - \frac{\Sigma^2}{A_2 + B_2} + N_1 + N_2}\Bigg). \end{aligned}$$

Now we simplify this maximization. The variables C_1, C_2 can be eliminated:

$$\sup_{\substack{A_2, B_1, B_2 \ge 0,\, \Sigma \in \mathbb{R} \\ \Sigma^2 \le B_1 B_2 \\ \Sigma^2 \le (P_1 - B_1)(P_2 - A_2 - B_2) \\ B_1 \le P_1 \\ A_2 + B_2 \le P_2}} g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma).$$

We can assume $B_1 B_2 \le (P_1 - B_1)(P_2 - A_2 - B_2)$ (otherwise we could increase the objective by increasing A_2 while fixing $A_2 + B_2$):

$$\sup_{\substack{A_2, B_1, B_2 \ge 0,\, \Sigma \in \mathbb{R} \\ \Sigma^2 \le B_1 B_2 \\ B_1 \le P_1 \\ A_2 + B_2 \le P_2 \\ B_1 B_2 \le (P_1 - B_1)(P_2 - A_2 - B_2)}} g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma).$$

We can further assume $B_1 B_2 = (P_1 - B_1)(P_2 - A_2 - B_2)$ (otherwise we would also have $A_2 + B_2 < P_2$ and we could increase the objective by increasing A_2 since the objective is increasing in A_2):

$$\sup_{\substack{A_2, B_1, B_2 \ge 0,\, \Sigma \in \mathbb{R} \\ \Sigma^2 \le B_1 B_2 \\ B_1 \le P_1 \\ A_2 + B_2 \le P_2 \\ B_1 B_2 = (P_1 - B_1)(P_2 - A_2 - B_2)}} g_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma).$$

If $B_1 \neq P_1$ then the constraint $B_1 B_2 = (P_1 - B_1)(P_2 - A_2 - B_2)$ means

$$A_2 = P_2 - \frac{P_1 B_2}{P_1 - B_1}. \tag{4.19}$$

If $B_1 = P_1$ then the constraints imply $B_2 = 0$ and $\Sigma = 0$, and the maximizer is given by $A_2 = P_2$. In both cases (4.19) is satisfied if $\frac{0}{0}$ is understood to be 0. From (4.19) the constraint $A_2 + B_2 \leq P_2$ is automatically satisfied and the constraint $A_2 \geq 0$ is equivalent to $\frac{B_1}{P_1} + \frac{B_2}{P_2} \leq 1$, which implies the constraint $B_1 \leq P_1$. So the maximization further simplifies to:

$$\sup_{\substack{B_1, B_2 \geq 0, \Sigma \in \mathbb{R} \\ \Sigma^2 \leq B_1 B_2 \\ \frac{B_1}{P_1} + \frac{B_2}{P_2} \leq 1}} g_{\lambda,\alpha}(P_2 - \frac{P_1 B_2}{P_1 - B_1}, B_1, B_2, \Sigma).$$

Finally, since replacing Σ by $|\Sigma|$ does not decrease the objective, we can assume $\Sigma \geq 0$:

$$\sup_{\substack{B_1, B_2, \Sigma \geq 0 \\ \Sigma \leq \sqrt{B_1 B_2} \\ \frac{B_1}{P_1} + \frac{B_2}{P_2} \leq 1}} g_{\lambda,\alpha}(P_2 - \frac{P_1 B_2}{P_1 - B_1}, B_1, B_2, \Sigma).$$

□

Proof of Proposition 4.7 (iii). It suffices to show that the maximization

$$\sup_{\substack{B_1, B_2 \geq 0 \\ \frac{B_1}{P_1} + \frac{B_2}{P_2} \leq 1}} \tilde{g}_{\lambda,\alpha}(P_2 - \frac{P_1 B_2}{P_1 - B_1}, B_1, B_2, \sqrt{B_1 B_2}), \tag{4.20}$$

is attained by $B_1 = B_2 = 0$, where the functional $\tilde{g}_{\lambda,\alpha}$ defined by

$$\begin{aligned}\tilde{g}_{\lambda,\alpha}(A_2, B_1, B_2, \Sigma) := \frac{1}{2}\Big((\lambda - 1)\alpha \log \frac{A_2 + N_1 + N_2}{N_1 + N_2} \\ + (\lambda - 1)(1 - \alpha) \log \frac{B_1 + B_2 + 2\Sigma + A_2 + N_1 + N_2}{B_1 + N_1} \\ + (1 + (\lambda - 1)(1 - \alpha)) \log \frac{B_1 + N_1}{B_1 + N_1 + N_2}\Big)\end{aligned}$$

is an upper bound to $g_{\lambda,\alpha}$.

The maximization (4.20) is attained by $B_1 = B_2 = 0$ if and only if for any B_1, B_2 satisfying the constraints we have

$$\begin{aligned}&\frac{1}{2}\Big((\lambda - 1)\alpha\left(\log\left(P_2 - \frac{P_1 B_2}{P_1 - B_1} + N_1 + N_2\right) - \log(N_1 + N_2)\right) \\ &\quad + (\lambda - 1)(1 - \alpha)\log\left(B_1 + B_2 + 2\sqrt{B_1 B_2} + P_2 - \frac{P_1 B_2}{P_1 - B_1} + N_1 + N_2\right) \\ &\quad + \log(B_1 + N_1) - (1 + (\lambda - 1)(1 - \alpha))\log(B_1 + N_1 + N_2)\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\Big((\lambda-1)\alpha\left(\log\left(P_2+N_1+N_2\right)-\log\left(N_1+N_2\right)\right)\\
&\quad+(\lambda-1)(1-\alpha)\log\left(P_2+N_1+N_2\right)\\
&\quad+\log\left(N_1\right)-(1+(\lambda-1)(1-\alpha))\log\left(N_1+N_2\right)\Big),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&(\lambda-1)\alpha\log\left(1-\frac{P_1B_2}{P_1-B_1}\frac{1}{P_2+N_1+N_2}\right)\\
&\quad+(\lambda-1)(1-\alpha)\log\left(1+\frac{\frac{B_1+2\sqrt{B_1B_2}-\frac{B_1B_2}{P_1-B_1}}{P_2+N_1+N_2}(N_1+N_2)-B_1}{B_1+N_1+N_2}\right)\\
&\quad+\log\left(1+\frac{B_1N_2}{(B_1+N_1+N_2)N_1}\right)\\
&\leq 0.
\end{aligned}
$$

By dividing the above inequality by λ and utilizing concavity of $x\mapsto\log(1+x)$, this is implied by

$$
\begin{aligned}
&-(\lambda-1)\alpha\frac{P_1B_2}{P_1-B_1}\frac{1}{P_2+N_1+N_2}\\
&\quad+(\lambda-1)(1-\alpha)\frac{\frac{B_1+2\sqrt{B_1B_2}-\frac{B_1B_2}{P_1-B_1}}{P_2+N_1+N_2}(N_1+N_2)-B_1}{B_1+N_1+N_2}+\frac{B_1N_2}{(B_1+N_1+N_2)N_1}\\
&\leq 0,
\end{aligned}
$$

or equivalently (by multiplying by $\frac{(B_1+N_1+N_2)(P_2+N_1+N_2)}{N_1+N_2}$)

$$
\begin{aligned}
&-(\lambda-1)\alpha\frac{P_1(B_1+N_1+N_2)}{(P_1-B_1)(N_1+N_2)}B_2\\
&\quad+(\lambda-1)(1-\alpha)\left(2\sqrt{B_1}\sqrt{B_2}-\frac{B_1}{P_1-B_1}B_2-\frac{B_1P_2}{N_1+N_2}\right)\\
&\quad+\frac{B_1N_2(P_2+N_1+N_2)}{N_1(N_1+N_2)}\\
&\leq 0.
\end{aligned}
$$

Since the left-hand side is quadratic in $\sqrt{B_2}$ with negative leading coefficient, this is implied by

$$
\begin{aligned}
&(\lambda-1)^2(1-\alpha)^2B_1+\left((\lambda-1)\alpha\frac{P_1(B_1+N_1+N_2)}{(P_1-B_1)(N_1+N_2)}+(\lambda-1)(1-\alpha)\frac{B_1}{P_1-B_1}\right)\\
&\quad\cdot\left(\frac{B_1N_2(P_2+N_1+N_2)}{N_1(N_1+N_2)}-(\lambda-1)(1-\alpha)\frac{B_1P_2}{N_1+N_2}\right)\\
&\leq 0,
\end{aligned}
$$

or equivalently (by multiplying by $\frac{P_1-B_1}{B_1(\lambda-1)}$)

$$\begin{aligned}
&(\lambda-1)(1-\alpha)^2(P_1-B_1)+\left(\alpha\left(P_1+\frac{P_1}{N_1+N_2}B_1\right)+(1-\alpha)B_1\right)\\
&\quad\cdot\left(\frac{N_2}{N_1}\left(1+\frac{P_2}{N_1+N_2}\right)-(\lambda-1)(1-\alpha)\frac{P_2}{N_1+N_2}\right)\\
&\leq 0.
\end{aligned}$$

Since the left-hand side is linear in B_1, this is true for all $B_1 \geq 0$ if the linear coefficient and the constant term are ≤ 0, that is

$$\begin{aligned}
&-(\lambda-1)(1-\alpha)^2+\left(\alpha\frac{P_1}{N_1+N_2}+(1-\alpha)\right)\\
&\quad\cdot\left(\frac{N_2}{N_1}\left(1+\frac{P_2}{N_1+N_2}\right)-(\lambda-1)(1-\alpha)\frac{P_2}{N_1+N_2}\right)\leq 0, \qquad (4.21)\\
&(\lambda-1)(1-\alpha)^2P_1\\
&\quad+\alpha P_1\left(\frac{N_2}{N_1}\left(1+\frac{P_2}{N_1+N_2}\right)-(\lambda-1)(1-\alpha)\frac{P_2}{N_1+N_2}\right)\leq 0. \qquad (4.22)
\end{aligned}$$

Inequality (4.22) implies

$$\frac{N_2}{N_1}\left(1+\frac{P_2}{N_1+N_2}\right)-(\lambda-1)(1-\alpha)\frac{P_2}{N_1+N_2}\leq 0,$$

which implies (4.21). Thus, it suffices to satisfy (4.22) since it implies (4.21). (4.22) is equivalent to

$$\frac{N_2}{N_1}\left(1+\frac{P_2}{N_1+N_2}\right)\leq(\lambda-1)\left((1-\alpha)\frac{P_2}{N_1+N_2}-\frac{(1-\alpha)^2}{\alpha}\right).$$

It can be shown using basic calculus that the right-hand side is ≥ 0 if and only if $\alpha \geq \frac{N_1+N_2}{P_2+N_1+N_2}$, and is maximized by $\alpha=\sqrt{\frac{N_1+N_2}{P_2+N_1+N_2}}$. Putting this maximizing α and rearranging we get

$$\lambda \geq 1+\frac{N_2}{N_1}\frac{1}{\left(1-\sqrt{\frac{N_1+N_2}{P_2+N_1+N_2}}\right)^2}=\lambda_0.$$

To conclude, if $\lambda \geq \lambda_0$ then letting $\alpha=\sqrt{\frac{N_1+N_2}{P_2+N_1+N_2}}$ gives that (4.20) is attained by $B_1=B_2=0$. Note that (4.20) upper bounds the maximization on right-hand side of (ii) and both objective functions are equal to each other when $B_1=B_2=0$ (in such case Σ must be 0). This completes the proof. □

Proof of Proposition 4.7 (iv). This is immediate from (ii) and (iii). □

Finally we have all the ingredients to prove Theorem 4.2.

Proof of Theorem 4.2. Denote

$$\begin{aligned}
\mathbf{Y}_1 &:= \mathbf{X}_1 + \mathbf{Z}_1, \\
\mathbf{T}_2 &:= \mathbf{X}_2 + \mathbf{Z}_1 + \mathbf{Z}_1, \\
\mathbf{Y}_2 &:= \mathbf{X}_2 + \mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2 = \mathbf{T}_2 + \mathbf{X}_1.
\end{aligned}$$

Then, for $\lambda \geq 1$ and $0 \leq \alpha \leq 1$,

$$\begin{aligned}
&(\lambda - 1)h(\mathbf{X}_2 + \mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2) + h(\mathbf{X}_1 + \mathbf{Z}_1) - \lambda h(\mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2) \\
&= (\lambda - 1)h(\mathbf{Y}_2) + h(\mathbf{Y}_1) - \lambda h(\mathbf{Y}_2|\mathbf{X}_2) \\
&= (\lambda - 1)\alpha I(\mathbf{X}_2; \mathbf{Y}_2) + (\lambda - 1)(1 - \alpha)\left(h(\mathbf{Y}_2) - h(\mathbf{Y}_1)\right) \\
&\qquad + (1 + (\lambda - 1)(1 - \alpha))\left(h(\mathbf{Y}_1|\mathbf{X}_2) - h(\mathbf{Y}_2|\mathbf{X}_2)\right) \\
&\overset{\text{(a)}}{\leq} (\lambda - 1)\alpha I(\mathbf{X}_2; \mathbf{T}_2) + (\lambda - 1)(1 - \alpha)\left(h(\mathbf{Y}_2) - h(\mathbf{Y}_1)\right) \\
&\qquad + (1 + (\lambda - 1)(1 - \alpha))\left(h(\mathbf{Y}_1|\mathbf{X}_2) - h(\mathbf{Y}_2|\mathbf{X}_2)\right) \\
&\overset{\text{(b)}}{\leq} \Theta_{\lambda,\alpha}(\mathbf{X}_1, \mathbf{X}_2) \\
&\overset{\text{(c)}}{\leq} G^{(n)}_{\lambda,\alpha}\left(\frac{\mathrm{E}[\|\mathbf{X}_1\|^2]}{n}, \frac{\mathrm{E}[\|\mathbf{X}_2\|^2]}{n}\right),
\end{aligned}$$

where (a) follows from data processing inequality, (b) follows from the definition of $\Theta_{\lambda,\alpha}$, and (c) follows from the definition of $G^{(n)}_{\lambda,\alpha}$, where $G^{(n)}_{\lambda,\alpha}$ is defined as in Proposition 4.7. Since α is arbitrary, we can take infimum over α and we have

$$\begin{aligned}
&(\lambda - 1)h(\mathbf{X}_2 + \mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2) + h(\mathbf{X}_1 + \mathbf{Z}_1) - \lambda h(\mathbf{X}_1 + \mathbf{Z}_1 + \mathbf{Z}_2) \\
&\leq \inf_{0 \leq \alpha \leq 1} G^{(n)}_{\lambda,\alpha}\left(\frac{\mathrm{E}[\|\mathbf{X}_1\|^2]}{n}, \frac{\mathrm{E}[\|\mathbf{X}_2\|^2]}{n}\right) \\
&\overset{\text{(d)}}{\leq} \frac{n}{2}\left((\lambda - 1)\log\left(\frac{1}{n}\mathrm{E}[\|\mathbf{X}_2\|^2] + N_1 + N_2\right) + \log N_1 - \lambda\log(N_1 + N_2)\right),
\end{aligned}$$

where (d) is a consequence of Proposition 4.7 (iv), where we use the condition $\lambda \geq \lambda_1$. □

Some lemmas

Here we state the lemmas that have been invoked in the proof of Theorem 4.2.

- Lemma 4.9 and 4.10 are required in the proof of Lemma 4.8 which justifies that the perturbed functional $\Theta^{(\epsilon,\delta)}_{\lambda,\alpha}$ converges uniformly and hence one can take limit of maximum value of the perturbed functional to obtain maximum value of the unperturbed functional $\Theta_{\lambda,\alpha}$. Lemma 4.9 is a statement of asymptotic bound on entropy by power. Lemma 4.10 allows interchanging limit and maximization under uniform convergence.

- Lemma 4.11, 4.12, 4.13 and 4.14 are used in the proof of Proposition 4.6 which gives an explicit expression for the maximization of $\Theta_{\lambda,\alpha}(\mathbf{X}_1, \mathbf{X}_2)$ over the space of Gaussian distributions.
- Lemma 4.15 gives a general condition for which the optimal value of an objective function is continuous under variation of the feasible set. Together wtih Proposition 4.5 (ii) which implies midpoint-concavity of $G^{(n)}_{\lambda,\alpha}(P_1, P_2)$, the maximum value of $\Theta_{\lambda,\alpha}(X_1^n, X_2^n)$ under power constraints P_1 and P_2, Lemma 4.15 establishes its continuity, and hence concavity and additivity as shown in Proposition 4.7 (i).

Lemma 4.9. *Let $\mathbf{Z}$ be a Gaussian random variable in $\mathbb{R}^d$ ($d \geq 1$) with an invertible covariance matrix. Then there exists $c \geq 0$ depending only on the covariance matrix of $\mathbf{Z}$ such that*

$$0 \leq h(\mathbf{X} + \mathbf{Z}|U) - h(\mathbf{Z}) \leq c \cdot \mathrm{E}[\|\mathbf{X}\|^2]$$

for any random variables $(\mathbf{X}, U) \perp \mathbf{Z}$ where $\mathbf{X}$ is in $\mathbb{R}^d$.

Proof. We have

$$\begin{aligned} h(\mathbf{X} + \mathbf{Z}|U) - h(\mathbf{Z}) &= h(\mathbf{X} + \mathbf{Z}|U) - h(\mathbf{X} + \mathbf{Z}|\mathbf{X}, U) \\ &= I(\mathbf{X}; \mathbf{X} + \mathbf{Z}|U) \\ &\geq 0. \end{aligned}$$

On the other hand, with

$$K := \mathrm{Cov}(\mathbf{Z})^{-1/2} \, \mathrm{Cov}(\mathbf{X}) \, \mathrm{Cov}(\mathbf{Z})^{-1/2},$$

we have

$$\begin{aligned} h(\mathbf{X} + \mathbf{Z}|U) - h(\mathbf{Z}) &\leq h(\mathbf{X} + \mathbf{Z}) - h(\mathbf{Z}) \\ &\overset{\text{(a)}}{\leq} \frac{1}{2} \log \left|2\pi e(\mathrm{Cov}(\mathbf{X}) + \mathrm{Cov}(\mathbf{Z}))\right| - \frac{1}{2} \log \left|2\pi e \, \mathrm{Cov}(\mathbf{Z})\right| \\ &= \frac{1}{2} \log |K + I| \\ &= \frac{1}{2} \sum_{i=1}^{d} \log \left(1 + \lambda_i(K)\right) \\ &\overset{\text{(b)}}{\leq} \frac{d}{2} \log \left(1 + \sum_{i=1}^{d} \frac{1}{d} \lambda_i(K)\right) \\ &= \frac{d}{2} \log \left(1 + \frac{1}{d} \mathrm{tr}(K)\right) \\ &\overset{\text{(c)}}{\leq} \frac{1}{2} \mathrm{tr}(K) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\operatorname{tr}(\operatorname{Cov}(\mathbf{X})\operatorname{Cov}(\mathbf{Z})^{-1}) \\
&\overset{\text{(d)}}{\leq} \frac{\operatorname{tr}(\operatorname{Cov}(\mathbf{X}))}{2\lambda_{\min}(\operatorname{Cov}(\mathbf{Z}))} \\
&\leq \frac{\mathrm{E}[\|\mathbf{X}\|^2]}{2\lambda_{\min}(\operatorname{Cov}(\mathbf{Z}))},
\end{aligned}
$$

where (a) holds since Gaussian maximizes entropy, (b) follows from Jensen's inequality, (c) holds since $\log(1+x) \leq x$ for $x \geq 0$, (d) follows from von Neumann's trace inequality, and $\lambda_i(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) denotes the i-th largest (respectively, the smallest) eigenvalue functional. $\square$

Lemma 4.10. *Let f and f_n $(n \geq 1)$ be real-valued functions defined on the same set. Suppose*

$$\lim_{n\to\infty} \sup_x |f_n(x) - f(x)| = 0.$$

Then

$$\lim_{n\to\infty} \sup_x f_n(x) = \sup_x f(x).$$

Proof. Let $\epsilon_n := \sup_x |f_n(x) - f(x)|$. Note that ϵ_n is bounded for sufficiently large n since ϵ_n converges. We have

$$f(x) - \epsilon_n \leq f_n(x) \leq f(x) + \epsilon_n$$

for any x and sufficiently large n. Taking supremum over x gives

$$\sup_x f(x) - \epsilon_n \leq \sup_x f_n(x) \leq \sup_x f(x) + \epsilon_n.$$

Then the result follows by squeezing. $\square$

Lemma 4.11. *Let $A, B, K, \tilde{K}$ be square matrices of same size such that $A \succeq B \succeq 0$ and $\tilde{K} \succeq K \succeq 0$. Then*

$$\log|K+B| - \log|K+A| \leq \log|\tilde{K}+B| - \log|\tilde{K}+A|.$$

Proof. The inequality is equivalent to

$$I(\mathbf{X}; \mathbf{X}+\mathbf{Z}) \geq I(\mathbf{X}; \mathbf{X}+\mathbf{Z}+\tilde{\mathbf{Z}})$$

for mutually independent Gaussian random variables $\mathbf{X} \sim \mathcal{N}(0, A-B)$, $\mathbf{Z} \sim \mathcal{N}(0, K+B)$ and $\tilde{\mathbf{Z}} \sim \mathcal{N}(0, \tilde{K}-K)$, which follows from the data processing inequality. $\square$

Lemma 4.12. *Let $U, \mathbf{X}, \mathbf{Z}_1, \mathbf{Z}_2$ be random variables where $\mathbf{X}$ is real-vector-valued, $\mathbf{Z}_1, \mathbf{Z}_2$ are Gaussians of the same dimension as $\mathbf{X}$, and $(U, \mathbf{X})$, $\mathbf{Z}_1$, $\mathbf{Z}_2$ are mutually independent. Then*

$$\begin{aligned} & h(\mathbf{X}+\mathbf{Z}_1|U) - h(\mathbf{X}+\mathbf{Z}_1+\mathbf{Z}_2|U) \\ & \leq \frac{1}{2}\left(\log\left|\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] + \mathrm{Cov}(\mathbf{Z}_1)\right| - \log\left|\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] + \mathrm{Cov}(\mathbf{Z}_1) + \mathrm{Cov}(\mathbf{Z}_2)\right|\right). \end{aligned}$$

Remark 4.3. *The scalar version of Lemma 4.12 is rather well-known. This is Exercise 9.21 in [CT91], for instance. It is also possible that the vector case is known but we present a short proof here for completeness.*

Proof. We have

$$\begin{aligned} & h(\mathbf{X}+\mathbf{Z}_1|U) - h(\mathbf{X}+\mathbf{Z}_1+\mathbf{Z}_2|U) \\ & \leq h(\mathbf{X} - \mathrm{E}[\mathbf{X}|U] + \mathbf{Z}_1|U) - h(\mathbf{X} - \mathrm{E}[\mathbf{X}|U] + \mathbf{Z}_1 + \mathbf{Z}_2|U) \\ & \leq \sup_{\substack{p(v,\tilde{\mathbf{x}}) \\ \mathrm{E}[\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T] \preceq \mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] \\ (V,\tilde{\mathbf{X}}) \perp (\mathbf{Z}_1,\mathbf{Z}_2)}} \left(h(\tilde{\mathbf{X}}+\mathbf{Z}_1|V) - h(\tilde{\mathbf{X}}+\mathbf{Z}_1+\mathbf{Z}_2|V)\right) \\ & \overset{\text{(a)}}{=} \sup_{0 \preceq K \preceq \mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)]} \frac{1}{2}\left(\log\left|K + \mathrm{Cov}(\mathbf{Z}_1)\right| - \log\left|K + \mathrm{Cov}(\mathbf{Z}_1) + \mathrm{Cov}(\mathbf{Z}_2)\right|\right) \\ & \overset{\text{(b)}}{=} \frac{1}{2}\left(\log\left|\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] + \mathrm{Cov}(\mathbf{Z}_1)\right| - \log\left|\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] + \mathrm{Cov}(\mathbf{Z}_1) + \mathrm{Cov}(\mathbf{Z}_2)\right|\right), \end{aligned}$$

where (a) is a consequence of Theorem 3.3, and (b) follows from Lemma 4.11. □

Lemma 4.13. *Let $U, \mathbf{X}$ be random variables where $\mathbf{X}$ is real-vector-valued. Then*

$$h(\mathbf{X}|U) \leq \frac{1}{2}\log\left|2\pi e\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)]\right|.$$

Proof. We have

$$\begin{aligned} h(\mathbf{X}|U) & \overset{\text{(a)}}{\leq} \mathrm{E}\left[\frac{1}{2}\log\left|2\pi e\,\mathrm{Cov}(\mathbf{X}|U)\right|\right] \\ & \overset{\text{(b)}}{\leq} \frac{1}{2}\log\left|2\pi e\,\mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)]\right|, \end{aligned}$$

where (a) holds since Gaussian maximizes entropy, and (b) follows from Jensen's inequality and concavity of log-determinant. □

Lemma 4.14. *Let $U, \mathbf{X}$ be random variables where $\mathbf{X}$ is real-vector-valued. Then*

$$\mathrm{Cov}(\mathbf{X}) - \mathrm{E}[\mathrm{Cov}(\mathbf{X}|U)] \succeq 0.$$

Proof. This follows from the law of total variance. □

Lemma 4.15. *Let f be a real-valued function defined on a metric space (X, d). Suppose f is Lipschitz, i.e., there exists a constant $C \geq 0$ such that*

$$|f(x) - f(y)| \leq C \cdot d(x, y)$$

for any $x, y \in X$. Suppose S and S_n $(n \geq 1)$ are subsets of X such that S_n converges to S in Hausdorff distance, i.e.,

$$\lim_n \max \left\{ \sup_{x \in S_n} \inf_{y \in S} d(x, y), \sup_{x \in S} \inf_{y \in S_n} d(x, y) \right\} = 0.$$

Then

$$\lim_n \sup_{x \in S_n} f(x) = \sup_{x \in S} f(x).$$

Proof. For any $x \in S_n$ and $y \in S$ it holds that

$$f(x) \leq f(y) + C \cdot d(x, y),$$

and hence

$$\sup_{x \in S_n} f(x) \leq \sup_{y \in S} f(y) + C \cdot \sup_{x \in S_n} \inf_{y \in S} d(x, y).$$

Taking limit superior yields

$$\limsup_n \sup_{x \in S_n} f(x) \leq \sup_{x \in S} f(x).$$

Similarly for any $x \in S$ and $y \in S_n$ it holds that

$$f(x) \leq f(y) + C \cdot d(x, y),$$

and hence

$$\sup_{x \in S} f(x) \leq \sup_{y \in S_n} f(y) + C \cdot \sup_{x \in S} \inf_{y \in S_n} d(x, y).$$

Taking limit inferior yields

$$\sup_{x \in S} f(x) \leq \liminf_n \sup_{x \in S_n} f(x).$$

This gives the result. $\square$

Chapter 5

Conclusion

The complete determination of capacity region of GIC has been a long-standing open problem in network information theory. The results in this thesis constitute a progress towards it.

In Chapter 2 we show that any multi-letter extensions to the Han–Kobayashi achievable region with Gaussian inputs for GIC coincide with the single-letter region. As a consequence, if one could show that the multi-letter extensions to the Han–Kobayashi achievable region is attained by Gaussian inputs, then the single-letter Han–Kobayashi achievable region with Gaussian inputs would be the capacity region.

In Chapter 3 we present two results. One of which is the computation of the slope of Han–Kobayashi achievable region with Gaussian inputs for GZIC at the Costa–Sato corner point. Another one of which concerns optimality of weighted sum-rates for a subclass of GIC with weak interference, establishing slope discontinuity at the maximum sum-rate point for the capacity region.

In Chapter 4 we propose a conjecture concerning Gaussian extremality of a functional, which would imply the optimality of Han–Kobayashi achievable region for GZIC. We then show an information inequality that establishes the conjecture in some regimes. The inequality also gives an outer bound for the slope of capacity region of GZIC at the Costa–Polyanskiy–Wu corner point.

Beyond the two-user case, for interference channel with additive Gaussian noise with three or more sender-receiver pairs it is known that structured codes that employ interference alignment [CJ08] outperform a natural generalization of the Han–Kobayashi scheme for the two receiver case. The results in this thesis suggest that for the two-user case, the Han–Kobayashi scheme with Gaussian signaling may also already be enough to be capacity-achieving.

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