On the AND-OR Interference Channel and the Sandglass Conjecture

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***Abstract*—This paper is mostly a follow up on the work of Etkin and Ordentlich that studied the capacity regions of binary input deterministic interference channels. The only binary input deterministic interference channel whose capacity region is unknown (up to isomorphism) is when one receiver receives the Boolean AND of the two transmitted symbols, and the other receiver obtains the Boolean OR of the two transmitted symbols. Etkin and Ordentlich stated in the paper that they believed that time-division would be the capacity region for this interference channel. In this paper we show that one can achieve rates outside the time-division region.**

That time-division yields the zero-error capacity region for this setting is known as Simonyi's sand-glass conjecture, a statement that has received considerable attention in the combinatorics community. Various upper-bounds on the sum-rate of the zero-error capacity region had been proposed in the combinatorics community. In this paper we evaluate an outer bound to the (traditional notion of) capacity region due to Etkin and Ordentlich and show that this yields, surprisingly, a tighter bound that the best known bound for the sand-glass problem.

Finally, we establish the capacity region of the some special classes of binary input interference channels by improving on the outer-bound proposed by Etkin and Ordentlich.

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I. Introduction

A two-sender two-receiver memoryless interference channel models a commonly occurring scenario of two point-to-point communications sharing a common medium and hence resulting in (possibly) mutual interference, see Figure I. An (n, R_1, R_2)-code for an interference channel consists of two encoders that map $[1 : \lfloor 2^{nR_1} \rfloor] \to \mathcal{X}_1^n$ and $[1 : \lfloor 2^{nR_2} \rfloor] \to \mathcal{X}_2^n$, respectively, and two decoders that map received sequences $\mathcal{Y}_1^n \to [1 : \lfloor 2^{nR_1} \rfloor]$ and $\mathcal{Y}_2^n \to [1 : \lfloor 2^{nR_2} \rfloor]$. The probability of error of a code (see Figure I for notation) is defined as $P_e := \mathrm{P}((\hat{M}_1, \hat{M}_2) \neq (M_1, M_2))$ where (M_1, M_2) is assumed to be uniformly distributed on $[1 : \lfloor 2^{nR_1} \rfloor] \times [1 : \lfloor 2^{nR_2} \rfloor]$. A non-negative rate pair (R_1, R_2) is said to be achievable if there exists a sequence of (n, R_1, R_2)-codes such that the probability of error tends to zero as n tends to infinity. The closure of the union of all achievable rate pairs is called the capacity region. Determining a computable characterization of the capacity region for this setting remains one of the central open problems in multi-user information theory. An inner bound proposed by Han and Kobayashi [1] has been shown to be sub-optimal, even for channels with binary input alphabets, in [2].

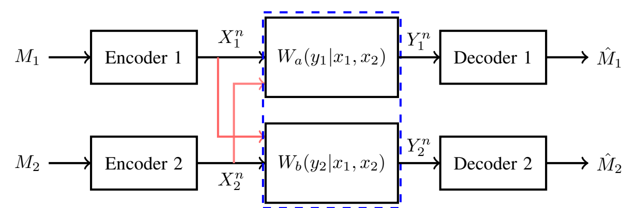

Fig. 1. Memoryless Interference Channel

A special case of binary interference channels where the channel outputs are deterministic functions of the input alphabets was studied by Etkin and Ordentlich in [3]. It is clear, via a strong interference argument, that if any of the receivers obtain an XOR of the two inputs then the sum-rate is upper bounded by 1 and that yields the capacity region by time-division. Therefore, upto isomorphism, there are only two interesting classes that needed to be addressed: (i) $Y_1 = X_1$ and $Y_2 = X_1 \vee X_2$ (the Boolean OR of the symbols), and (ii) $Y_1 = X_1 \vee X_2$ (Boolean AND) and $Y_2 = X_1 \wedge X_2$.

The authors developed the following upper bound to the capacity region for any discrete memoryless interference channel.

Theorem 1 (Theorem 1 in [3])**.** *Consider a discrete memoryless interference channel characterized by* $W_a(y_1|x_1, x_2)$ *and* $W_b(y_2|x_1, x_2)$. *The set of rate pairs* (R_1, R_2) *satisfying*

$$R_1 \leq \min\{I(X_1; Y_1|X_2, Q), I(U_2, X_1; Y_1|Q), H(X_1|U_1, Q)\}$$
$$R_2 \leq \min\{I(X_2; Y_2|X_1, Q), I(U_1, X_2; Y_2|Q), H(X_2|U_2, Q)\}$$

for some $p_Q p_{U_1,X_1|Q} p_{U_2,X_2|Q}$ *constitutes an outer bound to the capacity region of the interference channel.*

The authors showed that this outer bound was tight for the setting $Y_1 = X_1$ and $Y_2 = X_1 \vee X_2$ (see the last section for a generalization as well as a short self-contained proof).

The remaining setting, $Y_1 = X_1 \wedge X_2$ and $Y_2 = X_1 \vee X_2$ will be called the AND-OR interference channel (it was called IFC B in [3]) and this is the main focus of this article. It was conjectured that $R_1 + R_2 \leq 1$ is the capacity region for this setting (see pages 2601 and 2602 of [3]). They used Theorem

1 to deduce that any rate pair (R_1, R_2) that belongs to the capacity region must satisfy $R_1 + R_2 \leq 1.189....$

The two main contributions of this article with respect to the AND-OR interference channel are the following:

- We exhibit pairs (R_1, R_2) with $R_1 + R_2 > 1.01$ that lie inside the capacity region. This pair was computed by a careful evaluation of the Han and Kobayashi achievable region, which is a non-convex optimization problem.
- We improve the computation of the outer bound (another non-convex optimization problem) on the sum-rate for the AND-OR channel yielded by Theorem 1. We shows that any (R_1, R_2) that belongs to the capacity region must satisfy $R_1 + R_2 \leq 1.18031$ and further that there are rate pairs with $R_1 + R_2 \geq 1.18026$ that lie inside the bound given by Theorem 1. This upper bound, surprisingly, improves (albeit slightly) on even the upper bound for the zero-error version of this problem.

The zero-error code version of the AND-OR channel has received considerable attention in the combinatorics community and in the next section, we will summarize the bounds obtained there.

A. Zero-error capacity region and the sandglass conjecture

The following is a well-known conjecture (originally presented by Simonyi in an Oberwolfach conference in 1989) that appeared in [4].

Conjecture 1 ([4]). *Let $\mathcal{A}$ and $\mathcal{B}$ be set systems on an n element ground set. If for every $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ we have that*

$$A \cap B = A' \cap B' \implies A = A'$$
$$A \cup B = A' \cup B' \implies B = B'$$

then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Any pair of $\mathcal{A}, \mathcal{B}$ that satisfies the above two conditions are called a *recovering pair*.

Clearly by the canonical identification of the elements of $\mathcal{A}$ and $\mathcal{B}$ by sequences in $\{0,1\}^n$, we see that there is a 1-1 mapping between zero-error codes for the AND-OR interference channel and *recovering pairs*. Thus Conjecture 1 would imply that any rate pair (R_1, R_2) defined by a zero-error code for the AND-OR channel must satisfy $R_1 + R_2 \leq 1$.

An early non-trivial upper bound for Conjecture 1 was given by Holzman and Korner in [5]. They shows that if $\mathcal{A}, \mathcal{B}$ is a recovering pair then $|\mathcal{A}||\mathcal{B}| \leq (2.3264)^n$ or that $R_1 + R_2 \leq \log_2(2.3264) = 1.218$. This was later improved[1] by Soltész in [6] and it was show that any recovering pair must satisfy $|\mathcal{A}||\mathcal{B}| \leq (2.2814)^n$ or that $R_1 + R_2 \leq \log_2(2.2814) = 1.1899$. The work of Janzer in [7] improves the upper bound to $|\mathcal{A}||\mathcal{B}| \leq (2.2682)^n$ or that $R_1 + R_2 \leq \log_2(2.2682) = 1.1815$. Note that this bound is better than the bound of 1.189 using Theorem 1 computed by Etkin and Ordentlich. However we show that a careful computation of Theorem 1 yields an upper bound of 1.18031 which is an improvement even on the bound for zero-error capacity region.

[1]The work of Etkin and Ordentlich, which establishes a better bound, predates this work but since these were done in two 'essentially' non-interacting communities this was missed. One of the reasons for this article is to present the background work in both communities.

Notation: Throughout this paper, we employ the following notation $x_+ = \max\{x, 0\}$.

Remark 1. In the class of binary deterministic interference channels the zero-error capacity region is immediate, and matches the asymptotically vanishing error situation, except for the two settings: (i) $Y_1 = X_1$ and $Y_2 = X_1 \vee X_2$ and (ii) $Y_1 = X_1 \wedge X_2$ and $Y_2 = X_1 \vee X_2$. While, the capacity region of the latter is conjectured to be time-division; to the best of the knowledge of the authors the zero-error capacity region of the former remains an open problem. The asymptotic capacity region of this setting, as obtained in [3], is the intersection of $R_1 \leq 1$, and the collection of hyperplanes $R_1 + \lambda R_2 \leq \max_{0 \leq p,q \leq 1} \lambda H(pq) + (1 - \lambda q)_+ H(p)$, for all $\lambda \geq 1$.

The main ideas in the paper are related to computing the optimizers for non-convex optimization problems in information theory. Finally, we show a class of interference channels where a tailored outer bound that strictly improves on the Etkin-Ordentlich bound in Theorem 1 yields the capacity region.

II. The AND-OR interference channel

The main results of this section are the following.

Proposition 1. *For the AND-OR interference channel there exists achievable rate pairs (R_1, R_2) satisfying $R_1 + R_2 > 1.015$, demonstrating that time-division is not the capacity region.*

The proof of this proposition is found in Section II-A.

Proposition 2. *For the AND-OR interference channel there any achievable rate pair (R_1, R_2) must satisfy $R_1 + R_2 < 1.18031$, improving the existing outer bound for this channel as well as that for the sand-glass conjecture.*

The proof of this proposition is found in Section II-B.

A. Achievable rate pairs

The main contribution of this section is to produce a rate-pair (R_1, R_2) such that $R_1 + R_2 > 1$, thus disproving the conjecture in (see pages 2601 and 2602 of [3]) and hence establishing that if the sand-glass conjecture is correct, there is a gap between the asymptotic capacity region and the zero-error capacity region for this channel. On the other hand, it may also be possible that the construction of this rate pair might yield some insights into constructing potential counterexamples for the sand-glass conjecture.

From the achievable region proposed by Han and Kobayashi [1], it follows that any sum-rate $R_1 + R_2$ that satisfies

$$R_1 + R_2 \leq I(X_1; Y_1|U_2Q) + I(X_2; Y_2|U_1Q)$$
$$R_1 + R_2 \leq I(U_1X_2; Y_2|Q) + I(X_1; Y_1|U_1U_2Q)$$
$$R_1 + R_2 \leq I(U_2X_1; Y_1|Q) + I(X_2; Y_2|U_1U_2Q)$$
$$R_1 + R_2 \leq I(U_2X_1; Y_1|U_1Q) + I(U_1X_2; Y_2|U_2Q)$$

for some $p_Q p_{U_1,X_1|Q} p_{U_2,X_2|Q}$ is achievable. Note that this is obtained by performing Fourier-Motzkin elimination on the rate-region in Theorem 6.4 of [8].

In our channel setting we have $Y_1 = X_1 \wedge X_2$ and $Y_2 = X_1 \vee X_2$. We adopt the usual convention to denote $\bar{x} = 1 - x$. Let Q be a ternary random variable with $\mathrm{P}(Q = 0) = \bar{q}, \mathrm{P}(Q = 1) = \mathrm{P}(Q = 2) = \frac{q}{2}$. Conditioned on

$$\begin{aligned}
Q = 0: \quad & U_1 = X_1, U_2 = X_2, \mathrm{P}(X_1 = 0) = \mathrm{P}(X_2 = 1) = a, \\
Q = 1: \quad & U_1 = 1 \text{ (a constant)}, U_2 = X_2, \\
& \qquad \mathrm{P}(X_1 = 0) = b, \mathrm{P}(X_2 = 1) = c, \\
Q = 2: \quad & U_2 = 1 \text{ (a constant)}, U_1 = X_1, \\
& \qquad \mathrm{P}(X_1 = 0) = c, \mathrm{P}(X_2 = 1) = b.
\end{aligned}$$

Set $q = 0.9226, a = 0.3331, b = 0.4838, c = 0.9208$. With this choice, the four sum-rate constraints become

$$\begin{aligned}
R_1 + R_2 &\leq 2\bar{q}aH_2(a) + q\big(cH_2(b) + H_2(b\bar{c}) - \bar{c}H_2(b)\big) \\
&\qquad \approx 1.03958 \\
R_1 + R_2 &\leq \bar{q}H_2(a\bar{a}) + \frac{q}{2}\big(cH_2(b) + H_2(b\bar{c}) - \bar{c}H_2(b)\big) \\
&\qquad + \frac{q}{2}H_2(c\bar{b}) \approx 1.01575 \\
R_1 + R_2 &\leq \bar{q}H_2(a\bar{a}) + \frac{q}{2}\big(cH_2(b) + H_2(b\bar{c}) - \bar{c}H_2(b)\big) \\
&\qquad + \frac{q}{2}H_2(c\bar{b}) \approx 1.01575 \\
R_1 + R_2 &\leq 2\bar{q}\bar{a}H_2(a) + qH_2(c\bar{b}) \approx 1.01575,
\end{aligned}$$

where $H_2(x) = -x\log_2(x) - \bar{x}\log_2(\bar{x})$, denotes the binary entropy function. The values have been approximated to six significant figures. This shows that one can achieve a sum-rate pair $R_1 + R_2 \geq 1.015$, which contradicts the conjecture that time-division achieves the capacity under this setting.

Remark 2. Numerical search of Han-Kobayashi region is rather infeasible even for binary input interference channels. To obtain the above counterexample, we used a (lossless) symmetrization argument and other ideas to reduce the search space.

B. Outer bound to the capacity region

In this section we evaluate the outer bound for the sum-rate using Theorem 1. In particular Theorem 1 yields that and achievable rate pair must satisfy

$$\begin{aligned}
&R_1 + R_2 \\
&\leq \min_{\mu \in [0,1]} \max_{p_Q p_{U_1,X_1|Q} p_{U_2,X_2|Q}} \mu I(U_2X_1; Y_1|Q) + (1-\mu)H(X_1|U_1Q) \\
&\qquad + \mu I(U_1X_2; Y_2|Q) + (1-\mu)H(X_2|U_2Q) \\
&\overset{(a)}{=} \min_{\mu \in [0,1]} \max_{p_{U_1,X_1} p_{U_2,X_2}} \mu I(U_2X_1; Y_1) + (1-\mu)H(X_1|U_1) \\
&\qquad + \mu I(U_1X_2; Y_2) + (1-\mu)H(X_2|U_2) \\
&= \min_{\mu \in [0,1]} \max_{p_{X_1} p_{X_2}} \mu H(Y_1) + \mu H(Y_2) \\
&\qquad + \max_{p_{U_1|X_1}} \{(1-\mu)H(X_1|U_1) - \mu H(Y_2|U_1X_2)\} \\
&\qquad + \max_{p_{U_2|X_2}} \{(1-\mu)H(X_2|U_2) - \mu H(Y_1|U_2X_1)\} \\
&\overset{(b)}{=} \min_{\mu \in [0,1]} \max_{p_{X_1} p_{X_2}} \mu H(Y_1) + \mu H(Y_2) \\
&\qquad + \mathcal{C}_{p_{X_1}}[(1-\mu)H(X_1) - \mu H(Y_2|X_2)] \\
&\qquad + \mathcal{C}_{p_{X_2}}[(1-\mu)H(X_2) - \mu H(Y_1|X_1)] \\
&\overset{(c)}{=} \min_{\mu \in [0,1]} \max_{0 \leq p,q \leq 1} \mu H_2(p\bar{q}) + (1-\mu-\mu q)_+ H_2(p) \\
&\qquad + \mu H_2(\bar{p}q) + (1-\mu-\mu p)_+ H_2(q).
\end{aligned}$$

In the above (a) follows since conditioning on Q is computing an average over feasible distributions and hence while computing the maximum value we remove the averaging. The notation $\mathcal{C}_{p_X}[f]$ denotes the upper concave envelope of the functional $f(p_X)$ with respect to distributions on $\mathcal{X}$ and the equivalence follows from the link between maximization over auxiliaries and concave envelopes as demonstrated in [9]. Finally the equality in c follows since, say for the first concave envelope the inner function is either concave in p_{X_1} or convex in p_{X_1} depending on the sign of $1 - \mu - \mu q$.

Setting a particular choice of $\mu_0 = 0.720344$, we see that any achievable sum-rate satisfies

$$\begin{aligned}
R_1 + R_2 \leq \max_{0 \leq p,q \leq 1} & \mu_0 H_2(p\bar{q}) + (1-\mu_0-\mu_0 q)_+ H_2(p) \\
& + \mu_0 H_2(\bar{p}q) + (1-\mu_0-\mu_0 p)_+ H_2(q)
\end{aligned}$$

Using an interval arithmetic implementation in Julia we can obtain that the global maximum of this function over $(p,q) \in [0,1] \times [0,1]$ lies in the interval $[1.18026, 1.18031]$. Note that interval arithmetic implementations keep track of numerical errors and give a rigorous bound for the maximizer of elementary functions. This yields that any achievable rate pair (R_1, R_2) for the AND-OR interference channel must satisfy:

$$R_1 + R_2 \leq 1.18031.$$

As an immediate corollary, we obtain that if $(\mathcal{A}, \mathcal{B})$ is a *recovering pair* then

$$|\mathcal{A}||\mathcal{B}| \leq 2^{1.18031n} \approx (2.26625)^n.$$

This is an improvement over the best known bound for Simonyi's sand-glass conjecture.

A natural question to ask at this point is whether our computation of the outer bound provided by Theorem 1 can be improved to yield a significantly tighter bound. We now show that this is infeasible.

Let Q be a ternary random variable with $\mathrm{P}(Q = 0) = \frac{q}{2}, \mathrm{P}(Q = 1) = \frac{q}{2}$, $\mathrm{P}(Q = 2) = 1 - q$. Conditioned on

$$\begin{aligned}
Q = 0: \quad & U_1 = X_1, U_2 = 1 \text{ (a constant)}, \\
& \qquad \mathrm{P}(X_1 = 1) = u, \mathrm{P}(X_2 = 0) = v, \\
Q = 1: \quad & U_2 = X_2, U_1 = 1 \text{ (a constant)}, \\
& \qquad \mathrm{P}(X_1 = 1) = v, \mathrm{P}(X_2 = 0) = u, \\
Q = 2: \quad & U_1 = U_2 = 1 \text{ (a constant)}, \\
& \qquad \mathrm{P}(X_1 = 1) = \mathrm{P}(X_2 = 0) = a.
\end{aligned}$$

With this choice we obtain

$$
\begin{aligned}
H(X_i|U_i,Q) &= \frac{q}{2}H_2(v) + \bar{q}H_2(a), i=1,2\\
I(U_j,X_i;Y_i|Q) &= \frac{q}{2}(H_2(u\bar{v}) - uH_2(v) + H_2(v\bar{u})), i\neq j\\
&\quad + \bar{q}(H_2(a\bar{a}) - aH_2(a))\\
I(X_i;Y_i|X_j,Q) &= \frac{q}{2}(\bar{v}H_2(u) + \bar{u}H_2(v)) + \bar{q}\bar{a}H_2(a), i\neq j.
\end{aligned}
$$

Taking $q = 0.76034, a = 0.29897, u = 0.2557$, and $v = 0.4707$ we can see that the pairs $(R_1, R_2) = (0.59013, 0.59013)$ lies inside the region given by Theorem 1. This shows that the true sum-rate bound, S, yielded by Theorem 1 satisfies

$$1.18026 \leq S \leq 1.18031.$$

Remark 3. The computation was made possible using a min-max theorem in [10], symmetrization argument, and the identification of extremal auxiliaries with concave envelopes. Numerically the outer bound evaluates to about 1.180268 but to get this formal accuracy with interval arithmetic requires a lot of time.

III. Improved outer bound for a class of memoryless interference channels

The outer bound presented in Theorem 1 matches the capacity region for the interference channel described by: $Y_1 = X_1$ and $Y_2 = X_1 \vee X_2$. On the other hand, the presence of the channel independent entropy term $H(X_1|U_1,Q)$, which is an active constraint for the sum-rate bound above, is a strong indicator that the bound is not tight. The authors have been unable to replace that term with a mutual information term involving channel parameters for a generic interference channel. However for a special class listed below, such a replacement is possible and can be used to achieve the capacity region for some settings.

Definition 1. A *degraded Z-interference* channel is a memoryless interference channel such that $W_a(y_1|x_1,x_2) = W_a(y_1|x_1)$, i.e. the received symbol Y_1 is just a noisy version of X_1; and if there exists a channel $\hat{W}(y_2|x_2,y_1)$ such that $W_b(Y_2|x_1,x_2) = \sum_{y_1} W_a(y_1|x_1)\hat{W}(y_2|x_2,y_1)$.

Remark 4. This class of interference channels is interesting despite its degraded and one-sided interference structure. The sub-optimality of Han-Kobayashi achievable region [2] was demonstrated for a channel in this class. Further the scalar Gaussian Z-interference channel with weak interference, whose capacity has been open for decades, belongs to this class.

Theorem 2. *Consider a memoryless degraded Z-interference channel characterized by $W_a(y_1|x_1)$ and $\hat{W}(y_2|y_1,x_2)$. The set of rate pairs (R_1,R_2) satisfying*

$$
\begin{aligned}
R_1 &\leq I(X_1;Y_1|U_1,Q)\\
R_2 &\leq \min\{I(X_2;Y_2|X_1,Q), I(U_1,X_2;Y_2|Q)\}
\end{aligned}
$$

for some $p_Q p_{U_1,X_1|Q} p_{X_2|Q}$ constitutes an outer bound to the capacity region of the interference channel.

Proof. From standard arguments we have that

$$
\begin{aligned}
nR_1 &\leq I(X_1^n;Y_1^n) = \sum_{i=1}^n I(X_{1i};Y_{1i}|Y_1^{i-1})\\
nR_2 &\leq I(X_2^n;Y_2^n) = \sum_{i=1}^n I(X_2^n;Y_{2i}|Y_2^{i-1}) \leq \sum_{i=1}^n I(Y_2^{i-1},X_2^n;Y_{2i})\\
&\overset{(a)}{\leq} \sum_{i=1}^n I(Y_1^{i-1},X_{2i};Y_{2i}),
\end{aligned}
$$

where (a) follows since $Y_2^{i-1}, X_2^{n\setminus i} \to (Y_1^{i-1}, X_{2i}) \to Y_{2i}$ forms a Markov chain for a degraded Z-interference channel. Now identify $U_{1i} = Y_1^{i-1}$ and set Q to be uniform in $[1:n]$, to complete the argument in the standard manner. □

Note that the Han and Kobayashi achievable rate region for the degraded Z-interference channel reduces to the set of rate pairs (R_1,R_2) satisfying

$$
\begin{aligned}
R_1 &\leq I(X_1;Y_1|Q)\\
R_2 &\leq I(X_2;Y_2|U_1Q) \qquad (1)\\
R_1 + R_2 &\leq I(U_1X_2;Y_2|Q) + I(X_1;Y_1|U_1Q)
\end{aligned}
$$

for some $p_Q p_{U_1,X_1|Q} p_{X_2|Q}$.

Proposition 3. *For a degraded Z-interference channel the maximum achievable weighted sum-rate $C_\lambda := \lambda R_1 + R_2$ for $\lambda \geq 1$ is given by*

$$C_\lambda = \max_{p_{X_1}p_{X_2}} \lambda I(X_1;Y_1) + I(X_2;Y_2).$$

Proof. The outer bound in Theorem 2 yields that

$$
\begin{aligned}
C_\lambda &\leq \max_{p_Q p_{U_1,X_1|Q} p_{X_2|Q}} \lambda I(X_1;Y_1|U_1,Q) + I(U_1,X_2;Y_2|Q)\\
&= \max_{p_{U_1,X_1}p_{X_2}} \lambda I(X_1;Y_1|U_1) + I(U_1,X_2;Y_2)\\
&= \max_{p_{X_1}p_{X_2}} I(X_1,X_2;Y_2) + \mathcal{C}_{p_{X_1}}[\lambda I(X_1;Y_1) - I(X_1;Y_2|X_2)]\\
&\overset{(a)}{=} \max_{p_{X_1}p_{X_2}} I(X_1,X_2;Y_2) + \lambda I(X_1;Y_1) - I(X_1;Y_2|X_2)\\
&= \max_{p_{X_1}p_{X_2}} \lambda I(X_1;Y_1) + I(X_2;Y_2).
\end{aligned}
$$

In the above, the first equalities follow the same logic as the one in the beginning of Section II-B. The equality (a) follows as

$$\lambda I(X_1;Y_1) - I(X_1;Y_2|X_2) = (\lambda-1)I(X_1;Y_1) + I(X_1;Y_1|Y_2,X_2)$$

and hence is concave in p_{X_1} when $\lambda \geq 1$. The achievability of this weighted sum-rate is immediate by treating using the interference as noise decoding strategy. □

Consider the the following two degraded Z-interference channels. In both the channels the input alphabets are binary and $W_a(y_1|x_1)$ is a binary erasure channel with erasure probability ϵ_1.

- *Interference Channel Class A*: Here the receiver Y_2 receives the symbol $X_1 \vee X_2$ (the Boolean OR) passed through a binary erasure channel with erasure probability ϵ_2, with $\epsilon_2 \geq \epsilon_1$.

- *Interference Channel Class B*: Here the receiver Y_2 receives the symbol $X_1 \oplus X_2$ passed through a binary erasure channel with erasure probability ϵ_2, with $\epsilon_2 \geq \epsilon_1$.

To establish the capacity of these channels, it suffices (due to Proposition 3) to determine the maximum achievable weighted sum-rate $C_\lambda := \lambda R_1 + R_2$ for $\lambda < 1$.

Proposition 4. *For the Interference Channel Class A the maximum achievable weighted sum-rate $C_\lambda := \lambda R_1 + R_2$ for $\lambda \leq 1$ is given by*

$$C_\lambda = \max_{0 \leq p,q \leq 1} (1-\epsilon_2)H_2(pq) + (\lambda(1-\epsilon_1) - q(1-\epsilon_2))_+ H(p).$$

Proof. We start with the outer bound in Theorem 2.

$$\begin{aligned}
C_\lambda &\leq \max_{p_Q p_{U_1,X_1|Q} p_{X_2|Q}} \lambda I(X_1;Y_1|U_1,Q) + I(U_1,X_2;Y_2|Q) \\
&= \max_{p_{U_1,X_1} p_{X_2}} \lambda I(X_1;Y_1|U_1) + I(U_1,X_2;Y_2) \\
&= \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + \mathcal{C}_{p_{X_1}}[\lambda I(X_1;Y_1) - I(X_1;Y_2|X_2)] \\
&= \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + \mathcal{C}_{p_{X_1}}[(\lambda(1-\epsilon_1) \\
&\qquad - \mathrm{P}(X_2=0)(1-\epsilon_2))H(X_1)] \\
&\overset{(a)}{=} \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + \big(\lambda(1-\epsilon_1) \\
&\qquad - \mathrm{P}(X_2=0)(1-\epsilon_2)\big)_+ H(X_1) \\
&\overset{(b)}{=} \max_{0 \leq p,q \leq 1} (1-\epsilon_2)H_2(pq) + (\lambda(1-\epsilon_1) - q(1-\epsilon_2))_+ H(p).
\end{aligned}$$

Where (a) follows by observing that $cH(X)$ is concave in $p(X)$ when $c \geq 0$ and convex if $c \leq 0$. The equality (b) follows by setting $\mathrm{P}(X_1=0)=p, \mathrm{P}(X_2=0)=q$.

For a given λ, let p^*, q^* be a maximizer of

$$V_\lambda := \max_{0 \leq p,q \leq 1} H_2(pq) + (\lambda(1-\epsilon_1) - q(1-\epsilon_2))_+ H(p).$$

We know that $C_\lambda \leq V_\lambda$ from the previous argument.

Case 1: If $\lambda(1-\epsilon_1) - q^*(1-\epsilon_2) \geq 0$, note that considering the achievable rate pair (by treating interference as noise) $\lambda I(X_1;Y_1) + I(X_2;Y_2)$ with $\mathrm{P}(X_1=0)=p^*, \mathrm{P}(X_2=0)=q^*$ yields an achievable rate of

$$(1-\epsilon_2)H_2(p^*q^*) + (\lambda(1-\epsilon_1) - q^*(1-\epsilon_2))H(p^*)$$

thus showing that $V_\lambda \leq C_\lambda$.

Case 2: If $\lambda(1-\epsilon_1) - q^*(1-\epsilon_2) < 0$, then note that $V_\lambda \leq H_2(p^*q^*) \leq 1$ and hence $V_\lambda = C_\lambda = 1$. In particular the hyperplane passes through the point $(R_1,R_2) = (0,1)$. Thus in both cases the outer bound V_λ matches the achievable weighted sum-rate and we are done. □

Remark 5. If we extend the Interference Channel Class A by including the setting where $\epsilon_2 < \epsilon_1$, determining the capacity region remains an open problem, though in this case the channel is not a degraded Z-interference channel.

Proposition 5. *For the Interference Channel Class B the maximum achievable weighted sum-rate $C_\lambda := \lambda R_1 + R_2$ for $\lambda \leq 1$ is given by the time-division strategy, i.e.*

$$C_\lambda = \max\{\lambda(1-\epsilon_1), (1-\epsilon_2)\}.$$

Proof. Mimicking the first steps of the proof from the previous proposition we obtain

$$\begin{aligned}
C_\lambda &\leq \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + \mathcal{C}_{p_{X_1}}[\lambda I(X_1;Y_1) - I(X_1;Y_2|X_2)] \\
&= \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + \mathcal{C}_{p_{X_1}}[(\lambda(1-\epsilon_1) - (1-\epsilon_2))H(X_1)] \\
&\overset{(a)}{=} \max_{p_{X_1} p_{X_2}} I(X_1,X_2;Y_2) + (\lambda(1-\epsilon_1) - (1-\epsilon_2))_+ H(X_1) \\
&\overset{(b)}{=} \max_{0 \leq p,q \leq 1} (1-\epsilon_2)H_2(pq) + (\lambda(1-\epsilon_1) - (1-\epsilon_2))_+ H(p).
\end{aligned}$$

As before, let $V_\lambda := \max_{0 \leq p,q \leq 1} (1-\epsilon_2)H_2(pq) + (\lambda(1-\epsilon_1) - (1-\epsilon_2))_+ H(p)$. If $\lambda(1-\epsilon_1) \leq (1-\epsilon_2)$, then it is immediate that $V_\lambda = (1-\epsilon_2)$. If $\lambda(1-\epsilon_1) > (1-\epsilon_2)$, then

$$\begin{aligned}
V_\lambda &\leq \max_{0 \leq p,q \leq 1} (1-\epsilon_2)H_2(pq) + (\lambda(1-\epsilon_1) - (1-\epsilon_2))H(p) \\
&\leq (1-\epsilon_2) + (\lambda(1-\epsilon_1) - (1-\epsilon_2)) = \lambda(1-\epsilon_1).
\end{aligned}$$

where the last inequality follows by upper bounding each binary entropy term by one. This completes the proof. □

Remark 6. If we extend the Interference Channel Class B by including the setting where $\epsilon_2 < \epsilon_1$, determining the capacity region is not hard. In this case, there is strong interference at receiver Y_2, i.e. $I(X_1;Y_2|X_2) > I(X_1;Y_1|X_2)$ and hence we obtain another outer bound that $R_1 + R_2 \leq I(X_1,X_2;Y_2|Q)$. This combined with the usual constraints $R_1 \leq I(X_1 : Y_1|X_2,Q) = I(X_1;Y_1|Q)$ (since it is a Z-interference channel), and $R_2 \leq I(X_2;Y_2|X_1,Q)$ shows that joint decoding at receiver Y_2, yields the capacity region.

IV. Summary

We used the idea of identifying extremal auxiliaries using concave envelopes to improve the outer bound for the AND-OR interference channel and the new bound also improves on the best known bound for the sand-glass conjecture. We also show that the conjectured time-division region for the AND-OR interference channel is sub-optimal. Further, we determined the capacity region for some classes of degraded Z-interference channels by improving the outer bound in [3].
The authors wish to thank conversations and discussions on this problem with Profs. Janos Korner (who mentioned the sand-glass conjecture), Jaikumar Radhakrishnan, and Andrej Bogdanov. Mehdi Yazdanpanah contributed to this work while he was a PhD candidate at The Chinese University of Hong Kong. The work of Chandra Nair was partially supported by the following grants – GRF 14231916, GRF 14206518 – from RGC Hong Kong.

REFERENCES

[1] T. S. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” *Information Theory, IEEE Transactions on*, vol. 27, no. 1, pp. 49–60, jan 1981.

[2] C. Nair, L. Xia, and M. Yazdanpanah, “Sub-optimality of Han-Kobayashi achievable region for interference channels,” in *2015 IEEE International Symposium on Information Theory (ISIT)*, June 2015, pp. 2416–2420.

[3] R. H. Etkin and E. Ordentlich, “Analysis of deterministic binary interference channels via a general outer bound,” *IEEE Transactions on Information Theory*, vol. 57, no. 5, pp. 2597–2604, May 2011.

[4] R. Ahlswede and G. Simonyi, “On the optimal structure of recovering set pairs in lattices: the sandglass conjecture,” *Discrete Mathematics*, vol. 128, no. 1, pp. 389 – 394, 1994. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0012365X94901295

[5] R. Holzman and J. Körner, “Cancellative pairs of families of sets,” *European Journal of Combinatorics*, vol. 16, no. 3, pp. 263 – 266, 1995. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0195669895900314

[6] D. Soltész, “New bounds on simonyi’s conjecture,” *European Journal of Combinatorics*, vol. 70, pp. 251 – 267, 2018. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0195669818300052

[7] B. Janzer, “A new upper bound for cancellative pairs,” *Electronic Journal of Combinatorics*, vol. 25, 08 2017.

[8] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2012.

[9] C. Nair, “Upper concave envelopes and auxiliary random variables,” *International Journal of Advances in Engineering Sciences and Applied Mathematics*, vol. 5, no. 1, pp. 12–20, 2013. [Online]. Available: http://dx.doi.org/10.1007/s12572-013-0081-7

[10] Y. Geng, A. Gohari, C. Nair, and Y. Yu, “On Marton’s inner bound and its optimality for classes of product broadcast channels,” *IEEE Transactions on Information Theory*, vol. 60, no. 1, pp. 22–41, Jan 2014.