On the Gaussian Z-Interference channel

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Abstract

The optimality of Han-Kobayashi achievable region (with Gaussian signaling) remains an open problem for Gaussian interference channel. In this paper we focus on the Gaussian Z-interference channel. In this paper we first show that using correlated (over time) Gaussian signals do not improve on the Han-Kobayashi achievable rate region. Secondly we compute the slope of the Han and Kobayashi achievable region around the Sato's corner point and provides outer bounds to the slope.

I. Introduction

Gaussian interference channel is one of the most basic multiuser settings whose capacity region is as yet undetermined. The best known-achievable region for a two-receiver interference channel is due to Han and Kobayashi [7]. Recently it has been shown that there are two-receiver interference channels with discrete alphabets where Han and Kobayashi region is strictly inside the capacity region [11]. However for the Gaussian setting, the optimality of the Han and Kobayashi region (with Gaussian auxiliaries) remains an open challenge.

In the discrete memoryless setting it was shown that a 2-letter extension (coding in blocks of two symbols) of the Han-Kobayashi scheme strictly outperforms the single-letter (traditional) scheme. There has been some attempts, for instance see [8], at using correlated Gaussians to improve on the Han and Kobayashi scheme for the interference channel.

Remark 1. It is worth mentioning that the authors in [8] erroneously claim that the rates they achieve outperform state-of-the-art scheme. This is incorrect as they do not compare the rates to the Han and Kobayashi scheme with Gaussian signals and the use of time-sharing variable, Q, to do power control. It was known since the work of one of the authors [3] that using Q to do power control strictly improves the rate region. The sub-optimality of the region without power control can also be shown by using perturbations along Hermite polynomials [1].

In this paper, the first main result that is presented is a proof that correlated Gaussian signaling does not improve on the traditional scheme for the Gaussian interference channel. The second result that we show concerns evaluation of the slope of the Han and Kobayashi region around a corner point known as Sato's corner point.

A. Preliminaries

An interference channel is a model for communication where two point-to-point communications occur over a shared medium causing interference. The particular channel model that we study in this paper is called the Gaussian Z-interference setting, and the channel is described by:

$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_2 &= X_2 + aX_1 + Z_2. \end{aligned} \tag{1}$$

Here Z_1 and Z_2 are Gaussian variables, each distributed as $\mathcal{N}(0,1)$ and independent of X_1 and X_2. We further assume power constraints P_1, P_2 on X_1, X_2 and that $a \in (0,1)$. (Note that if $a = 0$ or $a \geq 1$, then the capacity region is fully determined; hence this regime is the only interesting case.)

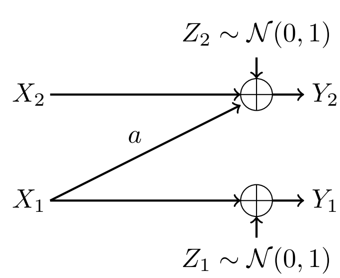

Fig. 1. Gaussian Z interference channel

The capacity region for this setting is defined in the usual sense (see [5] for details and background work).

The Han-Kobayashi achievable region [7] for the interference channel can be found in Section of [5]. However when the interference is one-sided (or Z), the the achievable region simplifies to the following.

Theorem 1 (Han-Kobayashi Region)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|Q) \\ R_2 &\leq I(X_2; Y_2|U_1, Q) \\ R_1 + R_2 &\leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q) \end{aligned}$$

over distributions $p_Q(q)p_{U_1,X_1|Q}(u_1, x_1|q)p_{X_2|Q}(x_2|q)$ *is achievable for a (discrete memoryless)* Z*-interference channel. To achieve this reqion, it suffices to consider* $|Q| \leq 3$.

The above region will also yield an achievable region in the Gaussian Z-interference channel defined by (1). Further, for every $Q = q$, if $X_1 = U_1 + V_1$, where U_1 and V_1 are zero-mean independent Gaussian random variables, and X_2 is also an independent Gaussian random variable, then we call such a region as *Han-Kobayashi region with Gaussian signaling*.

Theorem 2 (Han-Kobayashi region with Gaussian signaling)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + P_{1Q}\right)\right) \\ R_2 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + \frac{P_{2Q}}{1 + a^2\alpha_Q P_{1Q}}\right)\right) \\ R_1 + R_2 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + \alpha_Q P_{1Q}\right) + \frac{1}{2}\log\left(1 + \frac{P_{2Q} + a^2(1-\alpha_Q)P_{1Q}}{1 + a^2\alpha_Q P_{1Q}}\right)\right) \end{aligned}$$

over $\alpha_Q \in [0, 1], P_{1Q}, P_{2Q} \geq 0$ *satisfying* $\mathrm{E}_Q\left(P_{1Q}\right) \leq P_1$ *and* $\mathrm{E}_Q\left(P_{2Q}\right) \leq P_2$ *is achievable.*

By Bunt's extension of Caratheodory's theorem, it suffices to consider $|Q| \leq 5$. The region described by Theorem 2 will be referred to as $\mathcal{R}_{HK}$.

By computing the Han-Kobayashi region with Gaussian signaling of the multi-letter extension of the Gaussian interference channel the following region is achievable.

Theorem 3 (k-letter Han-Kobayashi region with Gaussian signaling)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq \frac{1}{k}\mathrm{E}_Q\left(\frac{1}{2}\log|I + K_{1Q}|\right) \\ R_2 &\leq \frac{1}{k}\mathrm{E}_Q\left(\frac{1}{2}\log\frac{|I + a^2K_{vQ} + K_{2Q}|}{|I + a^2K_{vQ}|}\right) \\ R_1 + R_2 &\leq \frac{1}{k}\mathrm{E}_Q\left(\frac{1}{2}\log|I + K_{vQ}| + \frac{1}{2}\log\frac{|I + a^2K_{1Q} + K_{2Q}|}{|I + a^2K_{vQ}|}\right) \end{aligned}$$

over $K_{1q}, K_{2q} \succeq 0, K_{vq} \preceq K_{1q}$ *satisfying* $\frac{1}{k}\mathrm{E}_Q(tr(K_{1Q})) \leq P_1$ *and* $\frac{1}{k}\mathrm{E}_Q(tr(K_{2Q})) \leq P_2$ *is achievable.*

By Bunt's extension of Caratheodory's theorem, it suffices to consider $|Q| \leq 5$. The $\succeq$ relation is used to denote the positive semi-definite partial order among real symmetric matrices; and $tr(A)$ denotes the trace of matrix A. The region described by Theorem 3 will be referred to as $\mathcal{R}_{HK}^{(k)}$. Clearly $\mathcal{R}_{HK} \subseteq \mathcal{R}_{HK}^{(k)} \subseteq \mathcal{C}$, the capacity region.

1) Known results about the capacity region: In this section we summarize the previously known results about the capacity region of the Gaussian Z-interference channel.

(i) It is known (from [3], [14], [13]) that the rate-pair $R_1 = \frac{1}{2}\log(1 + P_1)$ and $R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right)$, is a pareto-optimal point on the boundary of the capacity region. Further it is also known that the above point maximizes the rate-sum $R_1 + R_2$. Since $C_1 = \frac{1}{2}\log(1 + P_1)$ is the maximum achievable rate to receiver Y_1, the capacity region contains a line-segment that starts at $(C_1, 0)$ and ends at $(C_1, \frac{1}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right))$. We call this extremal point (corner point) as *Sato-point*.
The outer bounds in [13] and [9] shows that the *Sato-point* also maximizes $\lambda R_2 + R_1$ for any $\lambda \leq \frac{1+a^2P_1}{a^2(1+P_1)}$. This provides an outer bound to the slope of the capacity region around the Sato's corner point. We will recover this result in a self-contained manner for completeness.

(ii) It is known from (from [3], [12]) that the rate-pair $R_1 = \frac{1}{2}\log\left(1 + \frac{a^2P_1}{1+P_2}\right)$ and $R_2 = C_2 = \frac{1}{2}\log(1 + P_2)$, is another pareto-optimal point on the boundary of the capacity region. Hence the capacity region contains a line-segment that starts at $(0, C_2)$ and ends at $\left(\frac{1}{2}\log\left(1 + \frac{a^2P_1}{1+P_2}\right), \frac{1}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right)\right)$. This extremal point (corner point) is called in literature as *Costa-point*. The outer bound to the capacity region of the interference channel does not yield any finite λ such that the maximum of $\lambda R_2 + R_1$ over achievable rate pairs passes through the corner point. Two of the authors computed the slope of Han-Kobayashi region around the *Costa-point* [4].

B. Summary of our results

In this article we establish the following results.

Theorem 4. $\mathcal{R}_{HK}^{(k)} = \mathcal{R}_{HK}, \forall k \geq 1$.

We show that the k-letter extension of the Han-Kobayashi region with Gaussian signaling does not improve on the single-letter scheme.

Theorem 5. *The largest value of λ such that the maximum of $\lambda R_2 + R_1$ (with $(R_1, R_2) \in \mathcal{R}_{HK}$) occurs at the Sato point is given by*

$$\lambda_{cr} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \lambda^*\right\}$$

where λ^* is the unique positive solution of $h(\lambda^*) = 0$, where

$$\begin{aligned} h(\lambda) &:= \lambda\left(\log\left(1+\frac{P_2}{1+a^2P_1}\right) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right) \\ &\quad + \log\left(1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\lambda\right). \end{aligned}$$

II. Correlated Gaussians do not improve the region

In this section we establish Theorem 4. Towards this end we make the following observations: both $\mathcal{R}_{HK}^{(k)}$ and $\mathcal{R}_{HK}$ are convex regions, hence they can be characterized by the intersection of supporting hyperplanes. Further the hyperplane $R_1 + R_2$ (to the capacity region) passes through the Sato-point, which is present in both $\mathcal{R}_{HK}^{(k)}$ and $\mathcal{R}_{HK}$. Hence Theorem 4 is equivalent to showing that for every $\lambda > 1$,

$$\max_{(R_1,R_2)\in\mathcal{R}_{HK}^{(k)}} \lambda R_2 + R_1 = \max_{(R_1,R_2)\in\mathcal{R}_{HK}} \lambda R_2 + R_1.$$

The above condition can be re-expressed in terms of matrices as

$$\begin{aligned} &\max_{\substack{K_{1q},K_{2q}\succeq 0, K_{vq}\preceq K_{1q}: \\ \frac{1}{k}\mathrm{E}_Q(tr(K_{1Q}))\leq P_1, \frac{1}{k}\mathrm{E}_Q(tr(K_{2Q}))\leq P_2}} \mathrm{E}_Q\Big(\frac{1}{2k}\log|K_{2Q}+a^2K_{1Q}+I| + \frac{(\alpha-1)}{2k}\log|K_{2Q}+a^2K_{vQ}+I| \\ &\qquad + \frac{1}{2k}\log|K_{vQ}+I| - \frac{\alpha}{2k}\log|a^2K_{vQ}+I|\Big) \\ &= \max_{\substack{P_{1q},P_{2q}\succeq 0, \alpha_q\in[0,1]: \\ \mathrm{E}_Q(tr(P_{1Q}))\leq P_1, \mathrm{E}_Q(tr(P_{2Q}))\leq P_2}} \mathrm{E}_Q\Big(\frac{1}{2}\log|P_{2Q}+a^2P_{1Q}+1| + \frac{(\alpha-1)}{2}\log|P_{2Q}+a^2\alpha_QP_{1Q}+1| \\ &\qquad + \frac{1}{2}\log|\alpha_QP_{1Q}+1| - \frac{\alpha}{2}\log|a^2\alpha_QP_{1Q}+1|\Big). \end{aligned} \tag{2}$$

Identify $K_{uq} := K_{1q} - K_{vq}$. It is immediate that equation (2) will follow from Theorem 6 by the following reasoning: for every $Q = q$, Theorem 6 allows us to replace the matrices inside the expression by diagonal matrices, which then is just sum of terms of the form appearing on the right-hand-side. Note the extra factor $\frac{1}{k}$ on the left hand-side changes this sum into a convex combination and the equality is immediate.

Remark 2. The following notations will be used in the proof:

(i) Given a $k \times k$ matrix A, let $\lambda(A)$ denote the set (unordered) of eigenvalues of A, and let $\lambda^{\downarrow}(A)$ ($\lambda^{\uparrow}(A)$) denote the k-tuple of eigenvalues of A arranged in decreasing (increasing) order respectively.

(ii) Given two vectors v, w, we say $v \gg w$ if v *majorizes* w, i.e. if $v_{[1]} \geq v_{[2]} \cdots \geq v_{[k]}$ is a non-increasing arrangement of v and $w_{[1]} \geq w_{[2]} \cdots \geq w_{[k]}$ is a non-increasing arrangement of w; then

$$\sum_{l=1}^{m} v_{[l]} \geq \sum_{l=1}^{m} w_{[l]}, \quad 1 \leq m \leq k$$

with equality at $m = k$.

Theorem 6. *The maximum of the expression*

$$\begin{aligned} &\frac{1}{2}\log|K_2+a^2K_v+a^2K_u+I| + \frac{(\alpha-1)}{2}\log|K_2+a^2K_v+I| \\ &\quad + \frac{1}{2}\log|K_v+I| - \frac{\alpha}{2}\log|a^2K_v+I| \end{aligned}$$

where the $k \times k$ Hermitian matrices satisfy the constraints: $K_2, K_u, K_v \succeq 0, tr(K_u + K_v) \leq C_1, tr(K_2) \leq C_2$ can be attained by restricting K_2, K_u, K_v to be diagonal matrices.

Proof. Suppose we fix the matrices K_2 and K_v satisfying the trace constraint; then we must choose K_u so as to maximize $|K_2 + a^2 K_v + a^2 K_u + I|$ subject to $tr(K_u) \leq C_1 - tr(K_v)$.

If one further fixes the eigenvalues $\lambda(K_u)$ then Fiedler's bound [6] says that

$$|K_2 + a^2 K_v + a^2 K_u + I| \leq \prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u) \right),$$

and clearly equality is achieved if the matrices $K_2 + a^2 K_v + I$ and K_u share the same eigenvectors with eigenvalues $\lambda_i^{\downarrow}(K_2 + a^2 K_v + I)$ and $\lambda_i^{\uparrow}(K_u)$, respectively. Hence we seek to maximize

$$\prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u) \right),$$

subject to $\sum_{i=1}^{k} \lambda_i^{\uparrow}(K_u) \leq C_1 - tr(K_v)$ and $\lambda_i^{\uparrow}(K_u) \geq 0$. The optimal choice of this problem is the *water-filling solution*. Denote the optimal choice as K_u^w; then it is immediate that

$$\lambda_i^{\downarrow}(K_2 + a^2 K_v + I + a^2 K_u) = \lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u^w), \quad i = 1, .., k.$$

Let K_2^* be a diagonal matrix with entries ordered as $\lambda_i^{\downarrow}(K_2)$, and K_v^* be a diagonal matrix with entries ordered as $\lambda_i^{\uparrow}(K_v)$. Applying Fiedler's bound [6] we see that

$$|K_2 + a^2 K_v + I| \leq \prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2) + \lambda_i^{\uparrow}(K_v) + 1 \right) = |K_2^* + a^2 K_v^* + I|. \tag{3}$$

We now invoke the celebrated Lidskii-Weidlandt inequality (see (2.6) in survey article [2]) that establishes the majorization,

$$\lambda^{\downarrow}(K_2^* + a^2 K_v^* + I) = \lambda^{\downarrow}(K_2) + \lambda^{\uparrow}(a^2 K_v + I) \ll \lambda^{\downarrow}(K_2 + a^2 K_v + I).$$

Let $K_u^{*,w}$ denote the diagonal water filling matrix corresponding to $K_2^* + a^2 K_v^* + I$. Lemma 1 implies that

$$\lambda^{\downarrow}(K_2^* + a^2 K_v^* + I + K_u^{*,w}) \ll \lambda^{\downarrow}(K_2 + a^2 K_v + I + K_u^w).$$

Lemma 2 yields

$$|K_2^* + a^2 K_v^* + I + K_u^{*,w}| \geq |K_2 + a^2 K_v + I + K_u^w|. \tag{4}$$

Since $|K_v + I| = \prod_{i=1}^{k}(1 + \lambda_i(K_v))$ and $|a^2 K_v + I| = \prod_{i=1}^{k}(1 + a^2 \lambda_i(K_v))$, we see that for a fixed choice of $\lambda(K_v)$ and $\lambda(K_2)$, the diagonal matrices K_2^*, K_v^*, and $K_u^{*,w}$ maximize the expression (term-by-term) in Theorem 6. Varying over the choices of $\lambda(K_v)$ and $\lambda(K_2)$ that satisfies the trace constraint establishes the theorem. ☐

A. Lemmas regarding majorization and its applications

The following Lemma must be well-known but we cannot find an immediate reference, so we establish it below.

Lemma 1. *[Waterfilling preserves majorization] Let v, u be vectors such that $v \ll u$. Let v' and u' denote the vectors obtained after water-filling operation with a quantity of water $W > 0$. Then $v' \ll u'$.*

Proof. W.l.o.g. let v and u be arranged in non-increasing order. After waterfilling operation note that v' and u' is also in decreasing order; and further they satisfy

$$v_i' = \begin{cases} v_i & 1 \leq i \leq m \\ c & m+1 \leq i \leq k \end{cases},$$

$$u_i' = \begin{cases} u_i & 1 \leq i \leq n \\ c_1 & n+1 \leq i \leq k \end{cases}.$$

Further, $v_m \geq c \geq v_{m+1}$, $u_n \geq c_1 \geq u_{n+1}$, and $W = \sum_{i=m+1}^{k}(c - v_i) = \sum_{i=n+1}^{k}(c_1 - u_i)$.

We divide the proof into two cases: $m < n$ and $m \geq n$.
Case 1: $m < n$. Note that

$$\begin{aligned} &\sum_{i=1}^{k} v_i + W = \sum_{i=1}^{k} u_i + W \\ \implies &\sum_{i=1}^{m} v_i + (k-m)c = \sum_{i=1}^{n} u_i + (k-n)c_1 \geq \sum_{i=1}^{m} u_i + (k-m)c_1 \\ &\qquad\qquad\qquad\qquad\qquad \geq \sum_{i=1}^{m} v_i + (k-m)c_1. \end{aligned}$$

The first inequality is due to $u_i \geq c_1, m+1 \leq i \leq n$, and the second inequality is from $v \preceq u$. Hence $c \geq c_1$. Thus to establish that $v' \ll u'$, it suffices to show that

$$\sum_{i=1}^{l} v_i' \leq \sum_{i=1}^{l} u_i', \quad m+1 \leq i \leq n,$$

as the rest of the choices of l are immediate from $v \preceq u$, $c \geq c_1$, and that $\sum_{i=1}^{k} v_i' = \sum_{i=1}^{k} u_i'$.

We establish it by contradiction. Suppose l_o is the first index in $[m+1 : n]$ such that

$$\sum_{i=1}^{m} v_i + \sum_{i=m+1}^{l_o} c = \sum_{i=1}^{l_o} v_i' > \sum_{i=1}^{l_o} u_i' = \sum_{i=1}^{l_o} u_i.$$

Hence it must be that $c > u_{l_o} = u'_{l_o}$, and since u_i' is decreasing, $c \geq u_i', \ \forall i \geq l_o$. This would imply that

$$\sum_{i=1}^{k} v_i' = \sum_{i=1}^{l_o} v_i' + \sum_{i=l_o+1}^{k} c > \sum_{i=1}^{l_o} u_i' + \sum_{i=l_o+1}^{k} u_i' = \sum_{i=1}^{k} u_i',$$

a contradiction. This completes the proof in this case.

Case 2: $m \geq n$. Note that

$$W = \sum_{i=m+1}^{k} (c - v_i) = \sum_{i=n+1}^{k} (c_1 - u_i) \geq \sum_{i=m+1}^{k} (c_1 - u_i) \geq \sum_{i=m+1}^{k} (c_1 - v_i),$$

where the first inequality is due to $c - u_i \geq 0, \quad n+1 \leq i \leq m$, and the second inequality is from $v \ll u$, (tail of v larger larger partial sum than tail of u). Thus $c \geq c_1$, as before. Similar to previous case, to establish that $v' \ll u'$, it suffices to show that

$$\sum_{i=1}^{l} v_i' \leq \sum_{i=1}^{l} u_i', \quad n+1 \leq i \leq m,$$

as the rest of the choices of l are immediate from $v \preceq u$, $c \geq c_1$, and that $\sum_{i=1}^{k} v_i' = \sum_{i=1}^{k} u_i'$. The proof follows again by contradiction. Suppose l_o is the first index in $[n+1 : m]$ such that

$$\sum_{i=1}^{l_o} v_i = \sum_{i=1}^{l_o} v_i' > \sum_{i=1}^{l_0} u_i' = \sum_{i=1}^{n} u_i + \sum_{i=n+1}^{l_0} c_1.$$

Hence it must be that $v_{l_0} > c_1$, and since v_i' is decreasing, $v_i' \geq c_1, \ \forall i \geq l_0$. This would imply that

$$\sum_{i=1}^{k} v_i' = \sum_{i=1}^{l_o} v_i' + \sum_{i=l_o+1}^{k} v_i' > \sum_{i=1}^{l_o} u_i' + \sum_{i=l_o+1}^{k} c_1 = \sum_{i=1}^{k} u_i',$$

a contradiction. This completes the proof of the lemma. $\square$

Lemma 2. *(see A.1.d, page 166 in [10]) Let $A, B \prec 0$ and $\lambda(A) \preceq \lambda(B)$. Then $\prod_{i=1}^{k} \lambda_i(A) = |A| \ll |B| = \prod_{i=1}^{k} \lambda_i(B)$.*

It basically follows from the concavity of $\log(\cdot)$ and the Hardy-Littlewood-Polya majorization inequality.

III. Calculations

A. Outer Bound

B. Han-Kobayashi region with Gaussian signaling

For $\beta > 1$, the weighted sum-rate of the Han-Kobayashi region of a Z-interference channel can be computed as

$$\begin{aligned}
&\max_{R_1,R_2}(R_1+\beta R_2)\\
&=\max_{p(q)p_1(u_1x_1|q)p_2(x_2|q)}\{(\beta-1)I(X_2;Y_2|U_1Q)+I(U_1X_2;Y_2|Q)+I(X_1;Y_1|U_1Q)\}\\
&=\max_{p(q)p_1(u_1x_1|q)p_2(x_2|q)}\{I(X_1X_2;Y_2|Q)+(\beta-1)I(X_2;Y_2|U_1Q)-I(X_1;Y_2|U_1X_2Q)+I(X_1;Y_1|U_1Q)\}\\
&=\mathcal{C}_{X_1X_2}\{I(X_1X_2;Y_2)+\mathcal{C}_{X_1}\{(\beta-1)I(X_2;Y_2)-I(X_1;Y_2|X_2)+I(X_1;Y_1)\}\}
\end{aligned}$$

With Gaussian signaling, where $X_1=U_1+V_1$, $U_1\sim\mathcal{N}(0,(1-\alpha)Q_1)$, $V_1\sim\mathcal{N}(0,\alpha Q_1)$, $X_2\sim\mathcal{N}(0,Q_2)$ independent, the weighted sum-rate for power (P_1,P_2) is the concave envelope of $f_\beta(Q_1,Q_2)$ evaluated at (P_1,P_2), where f_β is defined by

$$f_\beta(Q_1,Q_2):=\frac{1}{2}\log(1+a^2Q_1+Q_2)+\max_{\alpha\in[0,1]}\left\{\frac{\beta}{2}\log\frac{1+a^2\alpha Q_1+Q_2}{1+a^2\alpha Q_1}+\frac{1}{2}\log\frac{1+\alpha Q_1}{1+a^2\alpha Q_1+Q_2}\right\}$$

By taking derivative with respect to α, the optimal $\alpha=\alpha^*$ satisfies:

$$\beta=\frac{1-a^2+Q_2}{a^2Q_2}\left(a^2+\frac{1-a^2}{1+\alpha^*Q_1}\right)$$

We write

$$\begin{aligned}
\mathcal{R}_1&=\left\{(Q_1,Q_2):\beta\ge\frac{1-a^2+Q_2}{a^2Q_2}\right\}\\
\mathcal{R}_2&=\left\{(Q_1,Q_2):\beta\le\frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)}\right\}\\
\mathcal{R}_3&=\left\{(Q_1,Q_2):\frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)}<\beta<\frac{1-a^2+Q_2}{a^2Q_2}\right\}
\end{aligned}$$

where $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3$ correspond to the cases $\alpha^*=0$, $\alpha^*=1$ and $0<\alpha^*<1$ respectively.

This gives explicit expressions for f_β

$$f_\beta(Q_1,Q_2)=\begin{cases}\frac{1}{2}\log(1+a^2Q_1+Q_2)+\frac{\beta-1}{2}\log(1+Q_2) & (Q_1,Q_2)\in\mathcal{R}_1\\ \frac{\beta}{2}\log(1+\frac{Q_2}{1+a^2Q_1})+\frac{1}{2}\log(1+Q_1) & (Q_1,Q_2)\in\mathcal{R}_2\\ \frac{1}{2}\log\frac{1+a^2Q_1+Q_2}{a^2Q_2}+\frac{\beta-1}{2}\log(\beta-1)-\frac{\beta}{2}\log\beta+\frac{\beta}{2}\log\frac{1-a^2+Q_2}{1-a^2}+\frac{1}{2}\log(1-a^2) & (Q_1,Q_2)\in\mathcal{R}_3\end{cases}$$

Now we compute the gradient and Hessian of f_β. In $\mathcal{R}_1$,

$$\begin{aligned}
\partial_{Q_1}f_\beta&=\frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2}\\
\partial_{Q_2}f_\beta&=\frac{1}{2}\frac{1}{1+a^2Q_1+Q_2}+\frac{\beta-1}{2}\frac{1}{1+Q_2}\\
\mathcal{H}f_\beta&=\begin{bmatrix}\frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2}\\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-1}{2}\frac{1}{(1+a^2Q_1+Q_2)^2}-\frac{\beta-1}{2}\frac{1}{(1+Q_2)^2}\end{bmatrix}
\end{aligned}$$

In $\mathcal{R}_2$,

$$\begin{aligned}
\partial_{Q_1}f_\beta&=\frac{1}{2}\left(\frac{a^2\beta}{1+a^2Q_1+Q_2}-\frac{a^2\beta}{1+a^2Q_1}+\frac{1}{1+Q_1}\right)\\
\partial_{Q_2}f_\beta&=\frac{\beta}{2}\frac{1}{1+a^2Q_1+Q_2}\\
\mathcal{H}f_\beta&=\begin{bmatrix}\frac{1}{2}\left(\frac{-a^4\beta}{(1+a^2Q_1+Q_2)^2}+\frac{a^4\beta}{(1+a^2Q_1)^2}-\frac{1}{(1+Q_1)^2}\right) & \frac{-a^2\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2}\\ \frac{-a^2\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2}\end{bmatrix}
\end{aligned}$$

In $\mathcal{R}_3$,

$$\partial_{Q_1} f_\beta = \frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2}$$
$$\partial_{Q_2} f_\beta = \frac{1}{2}\left(\frac{1}{1+a^2Q_1+Q_2} - \frac{1}{Q_2} + \frac{\beta}{1-a^2+Q_2}\right)$$
$$\mathcal{H} f_\beta = \begin{bmatrix} \frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{1}{2}\left(\frac{-1}{(1+a^2Q_1+Q_2)^2} + \frac{1}{Q_2^2} - \frac{\beta}{(1-a^2+Q_2)^2}\right) \end{bmatrix}$$

By checking the values f_β and ∇f_β at the boundaries, one can see that f_β is continuously differentiable on $\mathbb{R}^2_{>0}$.

C. Slope at the corner point

We would like to determine the largest β_{cr} such that the supporting hyperplane of the form $R_1 + \beta R_2$ passes through the corner point. That is,

$$\mathcal{C} f_\beta(P_1, P_2) = \frac{\beta}{2}\log(1 + \frac{P_2}{1+a^2P_1}) + \frac{1}{2}\log(1+P_1)$$

For β slightly larger than $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$, we have $(P_1, P_2) \in \mathcal{R}_3$. The function

$$\beta \mapsto f_\beta(P_1,P_2) - \left(\frac{\beta}{2}\log(1+\frac{P_2}{1+a^2P_1}) + \frac{1}{2}\log(1+P_1)\right)$$

$= 0$, derivative $= 0$ and second derivative > 0 at $\beta = \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$. So we have

$$\begin{aligned} \mathcal{C} f_\beta(P_1,P_2) &\geq f_\beta(P_1,P_2) \\ &> \frac{\beta}{2}\log(1+\frac{P_2}{1+a^2P_1}) + \frac{1}{2}\log(1+P_1) \end{aligned}$$

when β is slightly larger than $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$.

Hence we only need to consider β such that $(P_!, P_2) \in \mathcal{R}_2$. (Otherwise the supporting hyperplane of Han-Kobayashi region of the form $R_1 + \beta R_2$ will pass above the corner.) Then the hyperplane $R_1 + \beta R_2$ passes through the corner point if and only if $(P_1, P_2) \in \mathcal{R}_2$ and $\mathcal{C} f_\beta(P_1,P_2) = f_\beta(P_1,P_2)$.

Using the following lemma 3, this is equivalent to that $(P_1,P_2) \in \mathcal{R}_2$ and $g_\beta(Q_1,Q_2)$ attains global maximum at (P_1,P_2) for all $\beta \leq \beta_{cr}$, where g_β is defined by

$$g_\beta(Q_1,Q_2) := f_\beta(Q_1,Q_2) - \frac{1}{2}\left(\frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\right)Q_1 - \frac{1}{2}\left(\frac{\beta}{1+a^2P_1+P_2}\right)Q_2$$

Lemma 3. *Let f be a real-valued function differentiable at x. Then $\mathcal{C} f(x) = f(x)$ if and only if $f(\cdot) - \langle \nabla f(x), \cdot\rangle$ attains global maximum at x. Here $\mathcal{C} f$ and ∇f denotes the concave envelope and gradient of f, respectively.*

Proof. It suffices to show that $\mathcal{C} f(x) \leq f(x)$ if and only if for all h we have $f(x) \geq f(x+h) - \langle \nabla f(x), h\rangle$. The "if" part is immediate, by taking concave envelope with respect to h and then putting $h = 0$.

For the "only if" part, suppose on the contrary that there is $\epsilon > 0$ and $h \neq 0$ such that

$$f(x) + \epsilon \leq f(x+h) - \langle \nabla f(x), h\rangle$$

By differentiability of f at x,

$$|f(x+\zeta) - f(x) - \langle \nabla f(x), \zeta\rangle| \leq \frac{\epsilon}{2\|h\|}\|\zeta\|$$

for $\|\zeta\|$ small enough.

Now, for any $\delta \in (0,1)$,

$$\begin{aligned} f(x) &\geq \mathcal{C} f(x) \\ &\geq \delta \cdot \mathcal{C} f(x+h) + (1-\delta)\cdot \mathcal{C} f(x - \frac{\delta}{1-\delta}h) \\ &\geq \delta f(x+h) + (1-\delta) f(x - \frac{\delta}{1-\delta}h) \\ &\geq \delta\epsilon + \delta f(x) + \langle \nabla f(x), \delta h\rangle + (1-\delta) f(x - \frac{\delta}{1-\delta}h) \end{aligned}$$

Rearranging gives

$$\begin{aligned} f(x) &\geq \frac{\delta}{1-\delta}\epsilon + f(x - \frac{\delta}{1-\delta}h) - \left\langle \nabla f(x), -\frac{\delta}{1-\delta}h \right\rangle \\ &\geq \frac{\delta}{1-\delta}\epsilon + f(x) - \frac{\epsilon}{2\|h\|}\left\| -\frac{\delta}{1-\delta}h \right\| \\ &= f(x) + \frac{\epsilon}{2}\frac{\delta}{1-\delta} \end{aligned}$$

for δ small enough. This gives a contradiction.

□

We have the following result:

Theorem 7. *The maximum β such that the supporting hyperplane of the form $R_1 + \beta R_2$ of the Han-Kobayashi achievable region of the Gaussian Z-interference channel passes through the corner point $(R_1, R_2) = \left(\frac{1}{2}\log(1+P_1), \frac{1}{2}\log(1+\frac{P_2}{1+a^2P_1})\right)$ is given by*

$$\beta_{cr} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \beta^*\right\}$$

where β^* is the solution to $h(\beta^*) = 0$, with

$$h(\beta) := \beta\left\{\log(1+\frac{P_2}{1+a^2P_1}) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right\} + \log\left\{1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\beta\right\}$$

D. Interior analysis

In this part, we will show that g_β has local maximum in the interior only if $g_\beta(Q_1, Q_2) \leq g_\beta(P_1, P_2)$.

Claim 1.

$$\beta_{cr} \leq \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}$$

Proof. $\beta_{cr} \leq \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$ follows from the fact that $(P_1, P_2) \in \mathcal{R}_2$.

$\beta_{cr} \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$ follows from one of the second order conditions for (P_1, P_2) being a local maximum, namely $\det \mathcal{H} f_\beta(P_1, P_2) \geq 0$. □

Claim 2. *There is no local maximum of g_β in the interior of $\mathcal{R}_1$.*

Proof. Since g_β is concave in $\mathcal{R}_1$, there is at most one local maximum in the interior of $\mathcal{R}_1$. The first order condition reads

$$\begin{aligned} \frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2} &= \frac{1}{2}\left(\frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\right) \\ \frac{1}{2}\frac{1}{1+a^2Q_1+Q_2} + \frac{\beta-1}{2}\frac{1}{1+Q_2} &= \frac{1}{2}\left(\frac{\beta}{1+a^2P_1+P_2}\right) \end{aligned}$$

Solving for Q_2 gives

$$Q_2 = \frac{(1+a^2P_1)(1+P_1)}{1+P_1+\frac{1-\frac{1}{a^2}}{\beta-1}}$$

But in $\mathcal{R}_1$ we have $\beta \geq \frac{1-a^2+Q_2}{a^2Q_2}$. Substituting Q_2 gives

$$\beta \geq \frac{1+a^2P_1}{a^4(1+P_1)}$$

after some computation. But we also have $\beta \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, implying $a^2 \geq 1$. This gives a contradiction. □

Claim 3. *There are at most 2 local maxima of g_β in the interior of $\mathcal{R}_2$, both at which value $\leq g_\beta(P_1, P_2)$.*

Proof. The first order condition for g_β having a local maximum reads

$$\begin{aligned} \frac{a^2\beta}{1+a^2Q_1+Q_2} - \frac{a^2\beta}{1+a^2Q_1} + \frac{1}{1+Q_1} &= \frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1} \\ \frac{\beta}{1+a^2Q_1+Q_2} &= \frac{\beta}{1+a^2P_1+P_2} \end{aligned}$$

The solutions are

$$\begin{aligned} Q_1 &= P_1 \text{ or } \frac{\frac{1}{a^2}-1}{\frac{\beta}{k}-1}-1 \\ Q_2 &= P_2 + a^2(P_1 - Q_1) \end{aligned}$$

where $k := \frac{1+a^2P_1}{a^2(1+P_1)} \in (1, \beta]$. If (Q_1, Q_2) is one of the solutions, then

$$\begin{aligned} g_\beta(Q_1, Q_2) &= f_\beta(Q_1, Q_2) - \frac{\beta}{2}\frac{a^2Q_1 + Q_2}{1 + a^2Q_1 + Q_2} + \frac{\beta}{2}\frac{a^2Q_1}{1 + a^2Q_1} - \frac{1}{2}\frac{Q_1}{1 + Q_1} \\ &= f_\beta(Q_1, Q_2) + \frac{\beta}{2}\frac{1}{1 + a^2Q_1 + Q_2} - \frac{\beta}{2}\frac{1}{1 + a^2Q_1} + \frac{1}{2}\frac{1}{1 + Q_1} - \frac{1}{2} \\ &= \frac{\beta}{2}\varphi\left(1 + a^2Q_1 + Q_2\right) - \frac{\beta}{2}\varphi\left(1 + a^2Q_1\right) + \frac{1}{2}\varphi\left(1 + Q_1\right) - \frac{1}{2} \\ &= \frac{\beta}{2}\varphi\left(1 + a^2P_1 + P_2\right) - \frac{\beta}{2}\varphi\left(1 + a^2Q_1\right) + \frac{1}{2}\varphi\left(1 + Q_1\right) - \frac{1}{2} \end{aligned}$$

where $\varphi(x) := \log x + \frac{1}{x}$. Now let (Q_1, Q_2) to be the solution other than (P_1, P_2). Then,

$$\begin{aligned} g_\beta(P_1, P_2) - g_\beta(Q_1, Q_2) &= \frac{\beta}{2}\left(\varphi\left(1 + a^2Q_1\right) - \varphi\left(1 + a^2P_1\right)\right) - \frac{1}{2}\left(\varphi\left(1 + Q_1\right) - \varphi\left(1 + P_1\right)\right) \\ &= \frac{\beta}{2}\left(\varphi\left((1-a^2)\frac{\beta}{\beta - k}\right) - \varphi\left((1-a^2)\frac{k}{k-1}\right)\right) \\ &\quad - \frac{1}{2}\left(\varphi\left(\frac{1-a^2}{a^2}\frac{k}{\beta - k}\right) - \varphi\left(\frac{1-a^2}{a^2}\frac{1}{k-1}\right)\right) \end{aligned}$$

Differentiating with respect to β and simplifying gives

$$\partial_\beta\left(g_\beta(P_1, P_2) - g_\beta(Q_1, Q_2)\right) = \frac{1}{2}\left[\log\left(1 + \frac{k^2 - \beta}{k(\beta - k)}\right) - \frac{k^2 - \beta}{k(\beta - k)}\right] \leq 0$$

since $\beta \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 = k^2$ and $\forall x \geq 0, \log(1+x) \leq x$. So

$$g_\beta(P_1, P_2) - g_\beta(Q_1, Q_2) \geq g_{\beta=k^2}(P_1, P_2) - g_{\beta=k^2}(Q_1, Q_2) = 0$$

and hence $g_\beta(P_1, P_2) \geq g_\beta(Q_1, Q_2)$. $\square$

Claim 4. *There is no local maximum of g_β in the interior of $\mathcal{R}_3$.*

Proof. The first order condition for g_β having a local maximum reads

$$\begin{aligned} \frac{a^2}{1 + a^2Q_1 + Q_2} &= \frac{a^2\beta}{1 + a^2P_1 + P_2} - \frac{a^2\beta}{1 + a^2P_1} + \frac{1}{1 + P_1} \\ \frac{1}{1 + a^2Q_1 + Q_2} - \frac{1}{Q_2} + \frac{\beta}{1 - a^2 + Q_2} &= \frac{\beta}{1 + a^2P_1 + P_2} \end{aligned}$$

by substituting the first equation into the second one, and then writing $\beta = \frac{1-a^2+Q_2}{a^2Q_2}\theta$, where $\theta \in \left(\frac{1+a^2Q_1}{1+Q_1}, 1\right)$, we get

$$Q_2 = a^2(1 + P_1)$$

From the second order condition $\det \mathcal{H}f_\beta(Q_1, Q_2) \geq 0$, or equivalently,

$$\begin{aligned} \beta &\geq \left(\frac{1 - a^2 + Q_2}{Q_2}\right)^2 \\ &= \left(\frac{1 + a^2P_1}{a^2(1 + P_1)}\right)^2 \end{aligned}$$

which contradicts with that $\beta < \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$. $\square$

E. Boundary analysis

The remaining cases are the boundaries $Q_1 = 0$ and $Q_2 = 0$. In this part, we first establish that $g_\beta(P_1, P_2) \geq g_\beta(Q_1, Q_2)$ for (Q_1, Q_2) on the boundaries if and only if β is smaller than or equal to the upper bound in Claim 1 and β^* in Theorem 7. Then in Claim 8 we reduce the minimum of three terms to that of two of them. This gives the critical β in Theorem 7.

Claim 5. *On the boundary $Q_1 = 0$ we have that*

$$\min_{Q_2 \geq 0} \left(g_\beta(P_1, P_2) - g_\beta(0, Q_2)\right) \geq 0$$

if and only if $\beta \leq \frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1}$.

Proof. When $Q_1 = 0$, we have

$$\begin{aligned} f_\beta(Q_1, Q_2) &= \frac{\beta}{2}\log(1+Q_2) \\ g_\beta(Q_1, Q_2) &= \frac{\beta}{2}\log(1+Q_2) - \frac{\beta}{2}\frac{1}{1+a^2P_1+P_2}Q_2 \end{aligned}$$

$g_\beta(0, Q_2)$ is concave in Q_2, maximized at $Q_2 = a^2P_1 + P_2$. Note that $(P_1, P_2) \in \mathcal{R}_2$, we compute,

$$\begin{aligned} &\min_{Q_2 \geq 0} \left(g_\beta(P_1, P_2) - g_\beta(0, Q_2)\right) \\ &= g_\beta(P_1, P_2) - g_\beta(0, a^2P_1 + P_2) \\ &= -\frac{\beta}{2}\left(\log(1+a^2P_1) + \frac{1}{1+a^2P_1} - 1\right) + \frac{1}{2}\left(\log(1+P_1) + \frac{1}{1+P_1} - 1\right) \end{aligned}$$

≥ 0 if and only if $\beta \leq \frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1}$. □

Claim 6.

$$\frac{\log(1+P_1) + \frac{1}{1+P_1} - 1}{\log(1+a^2P_1) + \frac{1}{1+a^2P_1} - 1} \geq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$$

and hence, by Claim 1 and Claim5, $g_\beta(P_1, P_2) \geq g_\beta(Q_1, Q_2)$ on the boundary $Q_1 = 0$.

Proof. This is equivalent to

$$a^4\varphi(P_1) - \varphi\left(a^2P_1\right) \geq 0$$

where $\varphi(x) = (1+x)^2\log(1+x) - (1+x)x$. One can compute

$$\begin{aligned} \varphi'(x) &= 2(1+x)\log(1+x) - x \\ \varphi''(x) &= 2\log(1+x) + 1 \end{aligned}$$

Then

$$\begin{aligned} \partial_{P_1}\left(a^4\varphi(P_1) - \varphi\left(a^2P_1\right)\right) &= a^4\varphi'(P_1) - a^2\varphi'\left(a^2P_1\right) \\ \partial^2_{P_1}\left(a^4\varphi(P_1) - \varphi\left(a^2P_1\right)\right) &= a^4\left(\varphi''(P_1) - \varphi''\left(a^2P_1\right)\right) \\ &= a^4 \cdot 2\log\frac{1+P_1}{1+a^2P_1} \\ &\geq 0 \end{aligned}$$

So $\partial_{P_1}\left(a^4\varphi(P_1) - \varphi\left(a^2P_1\right)\right)$ is increasing in P_1 and hence $\geq (a^4 - a^2)\varphi'(0) = 0$. It follows that $a^4\varphi(P_1) - \varphi\left(a^2P_1\right)$ is also increasing in P_1 and hence $\geq (a^4 - 1)\varphi(0) = 0$. □

Claim 7. *On the boundary $Q_2 = 0$ we have that*

$$\min_{Q_1 \geq 0} \left(g_\beta(P_1, P_2) - g_\beta(Q_1, 0)\right) \geq 0$$

if and only if $\beta \leq \beta^*$, where β^* as in Theorem 7.

Proof. When $Q_2 = 0$, we have

$$\begin{aligned} f_\beta(Q_1, Q_2) &= \frac{1}{2}\log(1+Q_1) \\ g_\beta(Q_1, Q_2) &= \frac{1}{2}\log(1+Q_1) - \frac{1}{2}\left(\frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\right)Q_1 \end{aligned}$$

$g_\beta(Q_1,0)$ is concave in Q_1, maximized when

$$\begin{aligned}\frac{1}{1+Q_1} &= \frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\\ &\in [0,1] \text{ since } \beta \le \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}\end{aligned}$$

That is, there is always a maximizing $Q_1 \ge 0$. Note that $(P_1,P_2) \in \mathcal{R}_2$, after some computation,

$$\begin{aligned}&\min_{Q_1\ge 0}\left(g_\beta(P_1,P_2) - g_\beta(Q_1,0)\right)\\ &= \frac{1}{2}\left[\beta\left\{\log(1+\frac{P_2}{1+a^2P_1}) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right\} + \log\left\{1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\beta\right\}\right]\\ &= \frac{1}{2}h(\beta)\end{aligned}$$

which is concave in β, equals 0 when $\beta = 0$, the derivative with respect to β is non-negative at $\beta = 0$. Here $h(\cdot)$ as in Theorem 7. So it ≥ 0 if and only if $\beta \le \beta^*$. $\square$

Claim 8.

$$\min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \beta^*\right\} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2, \beta^*\right\}$$

where β^* as in Theorem 7.

Proof. It suffices to show that, if $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)} \ge \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, or equivalently $P_2 \le a^2(1+P_1)$, then $\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 \ge \beta^*$. That is, $h\left(\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right) \le 0$, where $h(\cdot)$ is defined in Theorem 7.

Write $P_2 = (1+a^2P_1)\theta$ and $k := \frac{1+a^2P_1}{a^2(1+P_1)}$. Then $\theta \le \frac{1}{k}$ and $k \ge 1$. We would like to show $h(k^2) \le 0$, that is,

$$\begin{aligned}& k^2\log(1+\theta) - k(k-1)\frac{\theta}{\theta+1} + \log(1-\frac{k\theta}{1+\theta}) \le 0\\ \Leftrightarrow \qquad & k^2\left(\log(1+\theta) + \frac{1}{1+\theta} - 1\right) + \left(\frac{k\theta}{1+\theta} + \log(1-\frac{k\theta}{1+\theta})\right) \le 0\end{aligned}$$

The derivative of left hand side with respect to θ is equal to

$$\begin{aligned}& k^2\frac{\theta}{(1+\theta)^2} + \frac{k}{(1+\theta)^2}\frac{k\theta}{k\theta-(1+\theta)}\\ &= \frac{k^2\theta}{(1+\theta)^2}\left(1 + \frac{1}{k\theta-(1+\theta)}\right)\\ &= \frac{k^2\theta^2}{(1+\theta)^2}\frac{1-k}{1+\theta-k\theta}\\ &\le 0\end{aligned}$$

So $k^2\left(\log(1+\theta) + \frac{1}{1+\theta} - 1\right) + \left(\frac{k\theta}{1+\theta} + \log(1-\frac{k\theta}{1+\theta})\right)$ is decreasing in θ, and $= 0$ when $\theta = 0$. We are done. $\square$

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