An enhanced genie-based outer bound for sum-rate of interference channels

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Abstract

We present a genie based outer bound for the sum rate of memoryless interference channels. When applied to the scalar Gaussian interference channel, this bound recovers all known capacity results.

1 Introduction and Background

Interference channel models the communication of two(or more) sender/receiver pairs with a shared medium for transmission. The interference channel shown below models the communication of two transmitter/receiver pairs and is the primary model we use for this study.

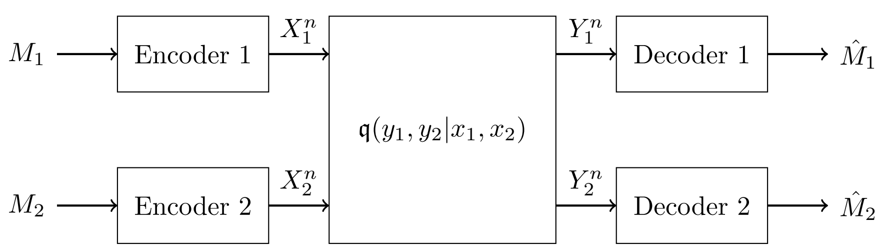

Fig. 1: Discrete memoryless interference channel

The *capacity region* is the closure of the set of achievable rate-pairs (R_1, R_2). For more background information on this problem and various definitions, please refer to Chapter 6 in [2].

The best known achievable region is described by Han-Kobayashi inner bound [4, 1]. It subsumes all other known (single-letter) inner bounds.

Theorem 1 (Han-Kobayashi (HK) inner bound)**.** *A rate-pair (R_1, R_2) is achievable for the channel described in Figure 1 if*

$$\begin{aligned}
R_1 &< I(X_1; Y_1|U_2, Q), \\
R_2 &< I(X_2; Y_2|U_1, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q), \\
R_1 + R_2 &< I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q),
\end{aligned}$$

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$$2R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q),$$
$$R_1 + 2R_2 < I(X_2, U_1; Y_2|Q) + I(X_2; Y_2|U_1, U_2, Q) + I(X_1, U_2; Y_1|U_1, Q)$$

for some distribution $p(q, u_1, u_2, x_1, x_2, y_1, y_2) = p(q)p(u_1, x_1|q)p(u_2, x_2|q)\mathfrak{q}(y_1, y_2|x_1, x_2)$ *with* $|U_1| \le |X_1|+4$, $|U_2| \le |X_2| + 4$, *and* $|Q| \le 7$.

It was not known until recently[1] whether HK inner bound is optimal. The two auxiliary random variables U_1, U_2 makes the evaluation of the bound impractical under most circumstances.

We restrict ourselves to the analysis of the sum-rate capacity $\mathcal{S} = \max_{R_1,R_2 \in \mathcal{C}}(R_1+R_2)$, where $\mathcal{C}$ denotes the capacity of the interference channel in Figure 1. A notion of *very weak interference*, which reduces HK inner bound on sum-rate to treating interference as noise, was defined in [5]. A genie based sum-rate outer bound was also derived in [5] and it was used to compute certain sum-capacity results. In this paper we extend this genie based outer bound to create an outer bound that recovers all known sum-capacity results in the Gaussian interference setting.

2 Previous results

In [5], the following results and definitions were obtained.

Definition 1 (Very weak interference)**.** An interference channel $\mathfrak{q}(y_1, y_2|x_1, x_2)$ is said to have *very weak interference* if

$$\begin{aligned} I(U_1; Y_1) &\ge I(U_1; Y_2|X_2), \\ I(U_2; Y_2) &\ge I(U_2; Y_1|X_1). \end{aligned} \tag{1}$$

for all auxiliaries (U_1, U_2) such that the joint probability distribution satisfies $p(u_1, u_2, x_1, x_2, y_1, y_2) = p_1(u_1, x_1)p_2(u_2, x_2)\mathfrak{q}(y_1, y_2|x_1, x_2)$.

Proposition 1. *The maximum achievable sum-rate of Han-Kobayashi inner bound, denoted as* $\mathcal{S}_{HK}(\mathfrak{q})$, *reduces to*

$$\mathcal{S}_{HK}(\mathfrak{q}) = \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2)$$

under very weak interference as defined in (1).

2.1 Binary Skewed Z Interference Channel (BSZIC)

We introduced a class of discrete memoryless interference channels with binary input/output. It was shown that this class of channels have very weak interference for a certain range of parameters.

Figure 2 depicts the transition probabilities $\mathfrak{q}(y_1|x_1, x_2)$, $\mathfrak{q}(y_2|x_1, x_2)$ of an interference channel. $p, q \in [0, 1]$ are constants. We call such a channel a *Binary Skewed-Z Interference Channel (BSZIC)*, or in this case, BSZIC(p, q).

Proposition 2. *A binary skewed-Z interference channel as shown in Figure 2 has very weak interference if and only if* $0 \le p + q \le 1$.

2.2 Genie-based Sum-rate Outer Bound

Theorem 2. *Let* T_1, T_2 *be any pair of random variables such that:* $p(y_1, t_1|x_1, x_2) = p(t_1|x_1)p(y_1|t_1, x_1, x_2)$, $p(y_2, t_2|x_1, x_2) = p(t_2|x_2)p(y_2|t_2, x_1, x_2)$, *and the marginals are consistent with the given channel transition probabilities, i.e.* $p(y_1|x_1, x_2) = \mathfrak{q}(y_1|x_1, x_2)$ *and* $p(y_2|x_1, x_2) = \mathfrak{q}(y_2|x_1, x_2)$. *The achievable sum-rate of the discrete memoryless interference channel characterized by* $\mathfrak{q}(y_1, y_2|x_1, x_2)$ *can be upper bounded as following:*

$$\begin{aligned} R_1 + R_2 \le \max_{p_1(x_1)p_2(x_2)} & I(X_1; T_1Y_1) + I(X_2; T_2Y_2) \\ & + \mathfrak{C}[I(X_2; T_2|X_1T_1) - I(X_2; Y_1|T_1X_1)] - I(X_2; T_2|X_1T_1) + I(X_2; Y_1|T_1X_1) \end{aligned} \tag{2}$$

[1] Recently it was shown by a subset of the authors, along with Babak Yazdanpanah, that the Han-Kobayashi inner bound is suboptimal. That study was a continuation of our efforts that started with this problem.

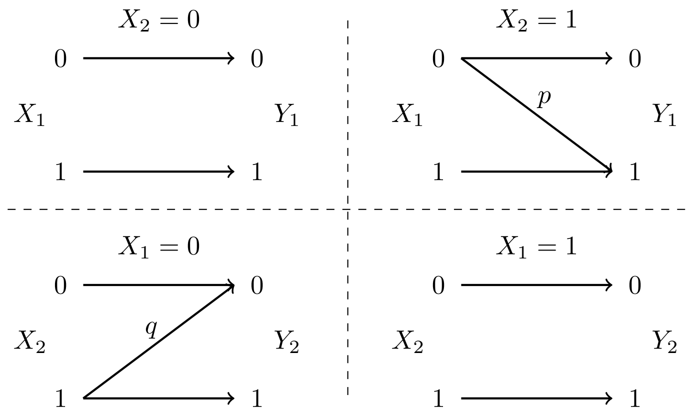

Fig. 2: Binary skewed-Z interference channel (BSZIC)

$$+\,\mathfrak{C}[I(X_1;T_1|X_2T_2) - I(X_1;Y_2|T_2X_2)] - I(X_1;T_1|X_2T_2) + I(X_1;Y_2|T_2X_2),$$

where $\mathfrak{C}[I(X_2;T_2|X_1T_1) - I(X_2;Y_1|T_1X_1)]$ *denotes the upper concave envelope of the function*[2] $I(X_2;T_2|X_1T_1) - I(X_2;Y_1|T_1X_1)$ *evaluated with respect to the space of product distributions* $p_1(x_1)p_2(x_2)$*. Similarly,* $\mathfrak{C}[I(X_1;T_1|X_2T_2) - I(X_1;Y_2|T_2X_2)]$ *denotes the upper concave envelope of the function* $I(X_1;T_1|X_2T_2) - I(X_1;Y_2|T_2X_2)$ *evaluated with respect to the same space of product distributions* $p_1(x_1)p_2(x_2)$*.*

By showing that the genie-based outer bound coincided with the inner bound the following result was established in [5].

Theorem 3 ([5])**.** *Treating interference as noise is sum-rate optimal for BSZIC when channel parameters* (p, q) *satisfy*

$$p + q + 3pq \leq 1.$$

The regime (as a subset of the very weak interference regime) is shown in Figure 3.

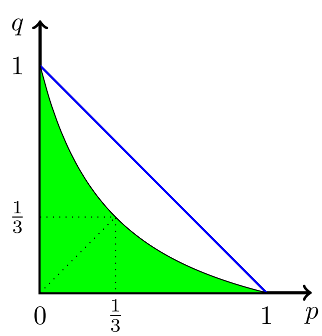

Fig. 3: Regime of parameters where sum-rate capacity is established for very weak BSZIC(p,q)

Proposition 3 ([5])**.** *For the binary skewed-Z interference channel when* $p = q = \frac{1}{2}$*, the genie based outer bound is strictly greater than treating interference as noise inner bound.*

Numerical simulations indicate that there the two-letter treating interference as noise sum-rate coincides with the one letter sum-rate even when $p = q = \frac{1}{2}$. This indicates a probable optimality of the Han-Kobayashi

[2] The upper concave envelope of a function $f(x)$ over domain $\mathcal{D}$ is defined as

$$\mathfrak{C}[f](x) := \inf\{g(x) : g(y) \text{ is concave in } \mathcal{D},\ g(y) \geq f(y)\ \forall y \in \mathcal{D}.\}.$$

sum-capacity for the entire regime of parameters of very weak interference. Thus we would like to improve the genie based outer bound.

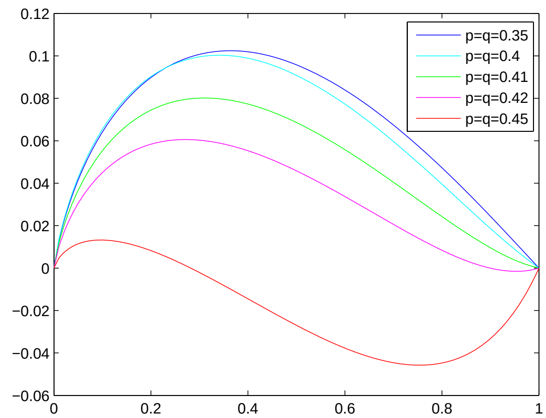

Fig. 4: Graph of $g_1(x)$ for different values of p.[1]

3 Enhanced Genie-based Sum-rate Outer Bound

Like all outer bounds, the genie-based outer bound is just a mathematical gadget for getting a single-letter expression that over estimates the rates of an encoding strategy. In contrast to usual outer bounds which involves maximization over auxiliary random variables (or equivalently, concave envelopes), the genie-based outer bound involves an additional minimization over the choice of genies. Such a gadget has been proven useful in obtaining the sum-rate capacity of certain discrete interference channels where said rate was not previously known. It also recovered the sum-rate result for a subset of the Gaussian weak interference regime. Since it involves minimizing over different genies along the process, the larger the space of genies, the better the outer bound could potentially be. We first state and prove an enhanced version of the genie-based sum-rate outer bound, which enlarges the space of genies to choose from as desired, for discrete memoryless interference channels, and then adapt the proof to obtain an outer bound for Gaussian channels where the codebooks have to satisfy a power constraint.

Theorem 4. *Let T_1, S_1, T_2, S_2 be any random variables satisfying:*

- $p(y_1, t_1, s_2|x_1, x_2) = p(t_1|x_1)p(s_2|x_2)p(y_1|t_1, s_2, x_1, x_2)$,
  $p(y_2, t_2, s_1|x_1, x_2) = p(t_2|x_2)p(s_1|x_1)p(y_2|t_2, s_1, x_1, x_2)$.
- *The marginals are consistent with the given channel transition probabilities, that is,* $p(y_1|x_1, x_2) = \mathfrak{q}(y_1|x_1, x_2)$ *and* $p(y_2|x_1, x_2) = \mathfrak{q}(y_2|x_1, x_2)$.
- *For each $k = 1, 2$, T_k, S_k has degraded order, i.e. either $X_k \to T_k \to S_k$ or $X_k \to S_k \to T_k$ must form a Markov chain.*

The achievable sum-rate, $SR = R_1 + R_2$, of the discrete memoryless interference channel characterized by $\mathfrak{q}(y_1, y_2|x_1, x_2)$ can be upper bounded as following:

$$
\begin{aligned}
SR \leq \max_{p_1(x_1)p_2(x_2)} \min_{T_1,S_1,T_2,S_2} \Big\{ & I(X_1; T_1, Y_1|S_2) + \mathcal{G}[I(X_1; T_1|X_2, T_2, S_1) - I(X_1; Y_2|X_2, T_2, S_1)] \\
& + I(X_2; T_2, Y_2|S_1) + \mathcal{G}[I(X_2; T_2|X_1, T_1, S_2) - I(X_2; Y_1|X_1, T_1, S_2)] \Big\},
\end{aligned} \tag{3}
$$

where $\mathcal{G}[\cdot]$ denotes the gap (difference) between the maximal convex combination of the function value over independent distributions, provided the same convex combination over the underlying independent distributions yields the independent distribution at which the value is being evaluated, and the function value. The minimum is taken over all genie variables (channels) that satisfy the conditions stated above.

Proof. The proof basically follows from Csiszar sum lemma and manipulations (using chain rule and data-processing inequality) of mutual information terms to reduce an n-letter expression to a 1-letter expression.

$$\begin{aligned}
nSR - n\epsilon &\le I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\
&\le I(X_1^n; Y_1^n T_1^n S_2^n) + I(X_2^n; Y_2^n T_2^n S_1^n) \\
&= I(X_1^n; T_1^n) + I(X_1^n; Y_1^n | T_1^n S_2^n) + I(X_2^n; T_2^n) + I(X_2^n; Y_2^n | T_2^n S_1^n) \\
&= \underline{H(T_1^n)} - H(T_1^n | X_1^n) + H(Y_1^n | T_1^n S_2^n) - \underline{H(Y_1^n | T_1^n S_2^n X_1^n)} \\
&\quad + \underline{H(T_2^n)} - H(T_2^n | X_2^n) + H(Y_2^n | T_2^n S_1^n) - \underline{H(Y_2^n | T_2^n S_1^n X_2^n)}
\end{aligned}$$

Note that

$$\begin{aligned}
H(T_1^n) - H(Y_2^n | T_2^n S_1^n X_2^n) &= H(T_1^n | S_1^n) + I(T_1^n; S_1^n) - H(Y_2^n | T_2^n S_1^n X_2^n) \\
&\overset{(a)}{=} H(T_1^n | T_2^n S_1^n X_2^n) + I(T_1^n; S_1^n) - H(Y_2^n | T_2^n S_1^n X_2^n) \\
&\overset{(b)}{=} \sum_i H(T_{1i} | T_1^{i-1} Y_{2,i+1}^n T_2^n S_1^n X_2^n) - H(Y_{2i} | T_1^{i-1} Y_{2,i+1}^n T_2^n S_1^n X_2^n) + I(T_1^n; S_1^n)
\end{aligned}$$

Equality (a) follows since (T_2^n, X_2^n) is independent of (T_1^n, S_1^n), while equality (b) is a consequence of Csiszar-sum identity.
Thus, we have

$$\begin{aligned}
&nSR - n\epsilon \\
&\quad \le \sum_i H(T_{1i} | T_1^{i-1} Y_{2,i+1}^n T_2^n S_1^n X_2^n) - H(Y_{2i} | T_1^{i-1} Y_{2,i+1}^n T_2^n S_1^n X_2^n) - H(T_{1i} | X_{1i}) + H(Y_{1i} | T_{1i} S_{2i}) \\
&\qquad + H(T_{2i} | T_2^{i-1} Y_{1,i+1}^n T_1^n S_2^n X_1^n) - H(Y_{1i} | T_2^{i-1} Y_{1,i+1}^n T_1^n S_2^n X_1^n) - H(T_{2i} | X_{2i}) + H(Y_{2i} | T_{2i} S_{1i}) \\
&\qquad + I(T_1^n; S_1^n) + I(T_2^n; S_2^n)
\end{aligned}$$

Using substitutions $U_{1i} = T_1^{i-1} S_1^{n \setminus i}$, $V_{1i} = X_2^{n \setminus i} T_2^{n \setminus i} Y_{2,i+1}^n$, $U_{2i} = T_2^{i-1} S_2^{n \setminus i}$, $V_{2i} = X_1^{n \setminus i} T_1^{n \setminus i} Y_{1,i+1}^n$, we have

$$\begin{aligned}
&nSR - n\epsilon \\
&\le \sum_i H(T_{1i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - H(Y_{2i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - \underline{H(T_{1i} | X_{1i} S_{1i})} - I(T_{1i}; S_{1i} | X_{1i}) + H(Y_{1i} | T_{1i} S_{2i}) \\
&\quad + H(T_{2i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - H(Y_{1i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - \underline{H(T_{2i} | X_{2i} S_{2i})} - I(T_{2i}; S_{2i} | X_{2i}) + H(Y_{2i} | T_{2i} S_{1i}) \\
&\quad + I(T_1^n; S_1^n) + I(T_2^n; S_2^n) \\
&\overset{(a)}{=} \sum_i H(T_{1i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - H(Y_{2i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - \underline{H(T_{1i} | X_{1i} U_{1i} V_{1i} T_{2i} S_{1i} X_{2i})} + H(Y_{1i} | T_{1i} S_{2i}) \\
&\quad + H(T_{2i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - H(Y_{1i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - \underline{H(T_{2i} | X_{2i} U_{2i} V_{2i} T_{1i} S_{2i} X_{1i})} + H(Y_{2i} | T_{2i} S_{1i}) \\
&\quad - I(T_{1i}; S_{1i} | X_{1i}) - I(T_{2i}; S_{2i} | X_{2i}) + I(T_1^n; S_1^n) + I(T_2^n; S_2^n) \\
&= \sum_i I(X_{1i}; T_{1i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - I(X_{1i}; Y_{2i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i}) - H(Y_{2i} | U_{1i} V_{1i} T_{2i} S_{1i} X_{2i} X_{1i}) + H(Y_{1i} | T_{1i} S_{2i}) \\
&\quad + I(X_{2i}; T_{2i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - I(X_{2i}; Y_{1i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i}) - H(Y_{1i} | U_{2i} V_{2i} T_{1i} S_{2i} X_{1i} X_{2i}) + H(Y_{2i} | T_{2i} S_{1i}) \\
&\quad - I(T_{1i}; S_{1i} | X_{1i}) - I(T_{2i}; S_{2i} | X_{2i}) + I(T_1^n; S_1^n) + I(T_2^n; S_2^n).
\end{aligned}$$

Equality (a) follows from the Markov chains

$$(U_{1i} V_{1i} T_{2i} X_{2i}) \to X_{1i} \to (T_{1i}, S_{1i}),\ (U_{2i} V_{2i} T_{1i} X_{1i}) \to X_{2i} \to (T_{2i}, S_{2i}).$$

Memoryless properties of the channels yield $U_{1i}, V_{1i} \to X_{2i}, X_{1i} \to T_{2i}, S_{1i}, Y_{2i}$ and $U_{2i}, V_{2i} \to X_{1i}, X_{2i} \to T_{1i}, S_{2i}, Y_{1i}$; hence

$nSR - n\epsilon$

$$
\begin{aligned}
&\le \sum_i I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - H(Y_{2i}|T_{2i}S_{1i}X_{2i}X_{1i}) + H(Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad + I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - H(Y_{1i}|T_{1i}S_{2i}X_{1i}X_{2i}) + H(Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad + I(X_{1i};Y_{1i}|T_{1i}S_{2i}) + I(X_{2i};Y_{2i}|T_{2i}S_{1i}) - I(T_{1i};S_{1i}|X_{1i}) - I(T_{2i};S_{2i}|X_{2i}) + I(T_1^n;S_1^n) + I(T_2^n;S_2^n) \\
&= \sum_i I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) + I(X_{2i};Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad + I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) + I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad + I(X_{1i};Y_{1i}|T_{1i}S_{2i}) + I(X_{2i};Y_{2i}|T_{2i}S_{1i}) - I(T_{1i};S_{1i}|X_{1i}) - I(T_{2i};S_{2i}|X_{2i}) + I(T_1^n;S_1^n) + I(T_2^n;S_2^n)
\end{aligned}
$$

When genies have degraded order, say $X_1 \to T_1 \to S_1$, we have

$$
\begin{aligned}
I(T_1^n;S_1^n) &= H(S_1^n) - H(S_1^n|T_1^n) \\
&\le \sum_i H(S_{1i}) - H(S_{1i}|S_1^{i-1}T_1^n) \\
&= \sum_i H(S_{1i}) - H(S_{1i}|T_{1i}) \\
&= \sum_i I(T_{1i};S_{1i})
\end{aligned}
$$

since $S_1^{i-1}T_1^{n\setminus i} \to X_{1i} \to T_{1i} \to S_{1i}$.

Thus, as long as the degradation order is consistent for every i, we have

$$
\begin{aligned}
&nSR - n\epsilon \\
&\le \sum_i I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) + I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad + I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};T_{2i}|T_{1i}S_{2i}X_{1i}) + I(X_{2i};Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad + I(X_{1i};T_{1i}|S_{1i}) + I(X_{2i};T_{2i}|S_{2i}) + I(X_{1i};Y_{1i}|T_{1i}S_{2i}) + I(X_{2i};Y_{2i}|T_{2i}S_{1i}) \\
&\quad - I(T_{1i};S_{1i}|X_{1i}) - I(T_{2i};S_{2i}|X_{2i}) + I(T_{1i};S_{1i}) + I(T_{2i};S_{2i}) \\
&= \sum_i I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) + I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad + I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};T_{2i}|T_{1i}S_{2i}X_{1i}) + I(X_{2i};Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad + I(X_{1i};T_{1i}) + I(X_{2i};T_{2i}) + I(X_{1i};Y_{1i}|T_{1i}S_{2i}) + I(X_{2i};Y_{2i}|T_{2i}S_{1i}) \\
&\overset{(a)}{=} \sum_i I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) + I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad + I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};T_{2i}|T_{1i}S_{2i}X_{1i}) + I(X_{2i};Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad + I(X_{1i};Y_{1i}T_{1i}|S_{2i}) + I(X_{2i};Y_{2i}T_{2i}|S_{1i})
\end{aligned}
$$

The equality (a) is due to (X_{1i}, T_{1i}) being independent of S_{2i}, and (X_{2i}, T_{2i}) being independent of S_{2i}. Further, it is an easy exercise to verify (say using d-separation principle) that

$$
X_{2i} \to (U_{1i}, V_{1i}) \to X_{1i}, \quad X_{2i} \to (U_{2i}, V_{2i}) \to X_{1i}
$$

are Markov chains. Hence

$$
\begin{aligned}
&I(X_{1i};T_{1i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|U_{1i}V_{1i}T_{2i}S_{1i}X_{2i}) - I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) + I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i}) \\
&\quad \le \mathcal{G}[I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i})], \\
&I(X_{2i};T_{2i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};Y_{1i}|U_{2i}V_{2i}T_{1i}S_{2i}X_{1i}) - I(X_{2i};T_{2i}|T_{1i}S_{2i}X_{1i}) + I(X_{2i};Y_{1i}|X_{1i}T_{1i}S_{2i}) \\
&\quad \le \mathcal{G}[I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i})].
\end{aligned}
$$

Since the above manipulations hold for every choice of valid genie channels (which could also depend on time-index i), we obtain

$$
\begin{aligned}
nSR - n\epsilon &\leq \sum_{i=1}^{n} \min_{T_{1i},S_{1i},T_{2i},S_{2i}} \Big\{ I(X_{1i};Y_{1i}T_{1i}|S_{2i}) + \mathcal{G}[I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i})] \\
&\qquad + I(X_{2i};Y_{2i}T_{2i}|S_{1i}) + \mathcal{G}[I(X_{1i};T_{1i}|T_{2i}S_{1i}X_{2i}) - I(X_{1i};Y_{2i}|X_{2i}T_{2i}S_{1i})] \Big\} \\
&\leq n \max_{p_1(x_1)p_2(x_2)} \min_{T_1,S_1,T_2,S_2} \Big\{ I(X_1;T_1,Y_1|S_2) + \mathcal{G}[I(X_1;T_1|X_2,T_2,S_1) - I(X_1;Y_2|X_2,T_2,S_1)]) \\
&\qquad + I(X_2;T_2,Y_2|S_1) + \mathcal{G}[I(X_2;T_2|X_1,T_1,S_2) - I(X_2;Y_1|X_1,T_1,S_2)] \Big\}.
\end{aligned}
$$

Taking $n \to \infty$ and $\epsilon \to 0$ yield the desired outer bound. □

Note that when $S_1 = S_2 = \emptyset$, enhanced genie-based sum-rate outer bound implies the bound in (2). The additional genies provide more freedom in searching for *good* genies but computation becomes more complicated. A caveat is that the two genies originating from each sender need to form a consistent degraded order.

3.1 Gaussian Interference Channel and Enhanced Genie-based Sum-rate Outer Bound

Consider a Gaussian interference channel,

$$
\begin{aligned}
Y_1 &= X_1 + bX_2 + Z_1, \\
Y_2 &= X_2 + aX_2 + Z_2,
\end{aligned}
$$

where X_1, X_2 are independent continues random variables with $E[X_1^2] \leq P_1$ and $E[X_2^2] \leq P_2$. Z_1, Z_2 are independent Gaussian noise $\mathcal{N}(0,1)$. $a, b \geq 0$ are constants.

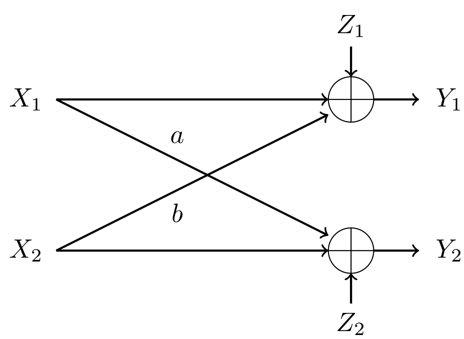

Fig. 5: Gaussian interference channel

Valid codebooks for a Gaussian interference channel need to satisfy individual power constraints. This allows one to write a slightly different outer bound than the one for discrete memoryless channels (with the same proof and assumptions on the genies). A similar outer bound can also be written for the discrete case, if one were to introduce type constraints on the codebooks. The proof of the following bound follows from the proof in the discrete case presented earlier and will not be repeated.

Theorem 5. *Let Q be any random variable taking finitely many values. For each choice $Q = q$, let $T_{1q}, S_{1q}, T_{2q}, S_{2q}$ be any random variables satisfying:*

- $p_q(y_1, t_1, s_2|x_1, x_2) = p_q(t_1|x_1)p_q(s_2|x_2)p_q(y_1|t_1, s_2, x_1, x_2)$,
  $p_q(y_2, t_2, s_1|x_1, x_2) = p_q(t_2|x_2)p_q(s_1|x_1)p_q(y_2|t_2, s_1, x_1, x_2)$.

- *The marginals are consistent with the given channel transition probabilities, that is,* $p_q(y_1|x_1,x_2) = \mathfrak{q}(y_1|x_1,x_2)$ *and* $p_q(y_2|x_1,x_2) = \mathfrak{q}(y_2|x_1,x_2)$.
- *For each* $k = 1,2$, T_{kq}, S_{kq} *has a consistent degraded order, i.e. either* $X_k \to T_{kq} \to S_{kq}, \forall q$ *or* $X_k \to S_{kq} \to T_{kq}$ $\forall q$ *holds.*

The achievable sum-rate, $SR = R_1 + R_2$, *of the Gaussian memoryless interference channel characterized by* $\mathfrak{q}(y_1,y_2|x_1,x_2)$ *can be upper bounded by the following:*

$$\max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \min_{T_{1q},S_{1q},T_{2q},S_{2q}} \Big\{ I(X_1;T_1,Y_1|S_2,Q) + \mathrm{E}_Q\, \mathcal{G}[I(X_1;T_1|X_2,T_2,S_1,q) - I(X_1;Y_2|X_2,T_2,S_1,q)]) \\ + I(X_2;T_2,Y_2|S_1,Q) + \mathrm{E}_Q\, \mathcal{G}[I(X_2;T_2|X_1,T_1,S_2,q) - I(X_2;Y_1|X_1,T_1,S_2,q)] \Big\}, \tag{4}$$

where $\mathcal{G}[\cdot]$ *denotes the gap (difference) between the maximal convex combination of the function value over independent distributions, provided the same convex combination over the underlying independent distributions yields the independent distribution at which the value is being evaluated, and the function value. The minimum is taken over all genie variables (channels) that satisfy the conditions stated above.*

The outer bound is not evaluable as is since there are no cardinality constraints on continuous input alphabets. However, since any valid choice of genies yields a valid outer bound, albeit loose might it be, we could still obtain computable outer bounds, just without guaranteed optimality. Even then, such optimality could be established if the outer bound value hits any known achievable rate value, which, with smart choices of genies, does happen for all cases where capacity was previously known. We shall proceed to show this.

In the sections below we use the standard notation that $C(x) = \frac{1}{2}\log(1+x)$. Further, we will just focus on establishing the outer bounds; the achievability of the sum-rates presented are already known in literature.

3.2 Optimality for Gaussian Interference Channel with Strong Interference

If $a, b \ge 1$, the Gaussian interference channel has strong interference. The sum-rate capacity is known to be

$$\mathcal{S} = \min\{C(P_1 + b^2P_2), C(P_2 + a^2P_1)\}.$$

Since $a \ge 1$, consider independent $\dot{Z}_1, \ddot{Z}_1 \sim \mathcal{N}(0,1)$ such that $Z_1 = \frac{1}{a}\dot{Z}_1 + \frac{\sqrt{a^2-1}}{a}\ddot{Z}_1$. Set $Y_1 = X_1 + bX_2 + \frac{1}{a}\dot{Z}_1 + \frac{\sqrt{a^2-1}}{a}\ddot{Z}_1$, $T_1 = X_1 + \frac{1}{a}\dot{Z}_1$, $S_1 = \emptyset$, $T_2 = \emptyset$ and $S_2 = X_2$.

Then note that for independent X_1, X_2 we have

$$I(X_1;T_1|X_2,T_2,S_1) = I(X_1;Y_2|X_2,T_2,S_1) \quad \Longrightarrow \mathcal{G}[I(X_1;T_1|X_2,T_2,S_1,q) - I(X_1;Y_2|X_2,T_2,S_1,q)]) = 0.$$

Similarly

$$0 = I(X_2;T_2|X_1,T_1,S_2) = I(X_2;Y_1|X_1,T_1,S_2) \quad \Longrightarrow \mathcal{G}[I(X_2;T_2|X_1,T_1,S_2,q) - I(X_2;Y_1|X_1,T_1,S_2,q)] = 0.$$

Thus, the genie-based sum-rate outer bound reduces to

$$\begin{aligned} SR &\le \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \Big\{ I(X_1;T_1,Y_1|S_2,Q) + +I(X_2;T_2,Y_2|S_1,Q) \Big\} \\ &\overset{(a)}{=} \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \Big\{ I(X_1;T_1|X_2,Q) + I(X_2;Y_2|Q) \Big\} \\ &\overset{(b)}{=} \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \Big\{ I(X_1;Y_2|X_2,Q) + I(X_2;Y_2|Q) \Big\} \\ &= \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} I(X_1,X_2;Y_2|Q) = C(P_2 + a^2P_1). \end{aligned}$$

In the above (a) follows from our various choices of genies and since $X_1 \to T_1 \to Y_1$ is Markov (by construction), and (b) follows by our choice of T_1. By symmetry, choosing another appropriate set of genies, we can get

$$SR \le C(P_1 + b^2 P_2).$$

Hence enhanced genie-based outer bound is tight for Gaussian interference channels with strong interference.

3.3 Optimality for Gaussian Interference Channel with Mixed Interference

If $a > 1$ and $b < 1$ (or $a < 1$ and $b > 1$), the Gaussian interference channel has mixed interference. For $a > 1$ and $b < 1$, the sum-rate capacity is known to be

$$SR = \min\left\{C\left(\frac{P_1}{b^2P_2+1}\right) + C\left(P_2\right), C(P_2 + a^2P_1)\right\}.$$

The bound $SR \le C(P_2 + a^2P_1)$ can be obtained by an identical selection of genies as in the previous section.

Since $b \le 1$, consider independent $Z_2, \dot{Z}_1 \sim \mathcal{N}(0,1)$ such that $Z_1 = bZ_2 + \sqrt{1-b^2}\dot{Z}_1$. Set $T_1 = \emptyset$, $S_1 = X_1$, $T_2 = X_2 + \frac{1}{b}Z_1 = X_2 + Z_2 + \frac{1-b^2}{b}\dot{Z}_1$, $S_2 = \emptyset$ and observe that for independent X_1, X_2 we have

$$0 = I(X_1;T_1|X_2,T_2,S_1) = I(X_1;Y_2|X_2,T_2,S_1) \quad \implies \mathcal{G}[I(X_1;T_1|X_2,T_2,S_1,q) - I(X_1;Y_2|X_2,T_2,S_1,q)]) = 0,$$

$$I(X_2;T_2|X_1,T_1,S_2) = I(X_2;Y_1|X_1,T_1,S_2) \quad \implies \mathcal{G}[I(X_2;T_2|X_1,T_1,S_2,q) - I(X_2;Y_1|X_1,T_1,S_2,q)] = 0.$$

Thus, the genie-based sum-rate outer bound reduces to

$$\begin{aligned}
SR &\le \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \left\{I(X_1;T_1,Y_1|S_2,Q) + I(X_2;T_2,Y_2|S_1,Q)\right\}\\
&= \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \left\{I(X_1;Y_1|Q) + I(X_2;T_2,Y_2|X_1,Q)\right\}\\
&\overset{(a)}{=} \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \left\{h(Y_1|Q) - h(bX_2 + Z_1|Q) + I(X_2;Y_2|X_1,Q)\right\}\\
&= \max_{\substack{Q,X_1,X_2\\ E(X_1^2)\le P_1, \mathrm{E}(X_2^2)\le P_2}} \left\{h(X_1 + bX_2 + Z_1|Q) - h(bX_2 + Z_1|Q) + h(X_2 + Z_2|Q) - h(Z_2|Q)\right\}\\
&\overset{(b)}{\le} C\left(\frac{P_1}{b^2P_2+1}\right) + C\left(P_2\right).
\end{aligned}$$

Here (a) holds since $I(X_2;T_2|Y_2,X_1,Q) = 0$, and (c) follows from a standard application of EPI. Hence enhanced genie-based sum-rate outer bound is tight for Gaussian interference channels with mixed interference.

3.4 Optimality for Gaussian Interference Channel in subset of Weak Interference regime

If $a, b < 1$, the sum-capacity of Gaussian interference channel is generally unknown. The sum-capacity is known when $a, b < 1$ and $a(b^2P_2 + 1) + b(a^2P_1 + 1) \le 1$. The sum-rate capacity in this case is

$$SR = C\left(\frac{P_1}{b^2P_2+1}\right) + C\left(\frac{P_2}{a^2P_1+1}\right).$$

Enhanced genie-based sum-rate outer bound turns out to be tight in this regime, too. Here we are really mimicking the argument in [2], albeit in single-letter.

Consider independent $\dot{Z}_1, \ddot{Z}_1 \sim \mathcal{N}(0,1)$ and let $Z_1 = \rho_1\dot{Z}_1 + \sqrt{1-\rho_1^2}\ddot{Z}_1$. Similarly consider independent $\dot{Z}_2, \ddot{Z}_2 \sim \mathcal{N}(0,1)$ and let $Z_2 = \rho_2\dot{Z}_2 + \sqrt{1-\rho_2^2}\ddot{Z}_2$, where $|\rho_1|, |\rho_2| \le 1$. Set $S_1 = S_2 = \emptyset$ and let $T_1 = X_1 + \eta_1\dot{Z}_1, T_2 = X_2 + \eta_2 Z_2$. Further let the parameters satisfy the following conditions:

- $\rho_1\eta_1 = b^2P_2 + 1$, $\rho_2\eta_2 = a^2P_1 + 1$
- $a^2\eta_1^2 \le 1 - \rho_2^2$, $b^2\eta_2^2 \le 1 - \rho_2^2$.

An easy exercise verifies that parameters satisfying these assumptions exist precisely when $a(b^2P_2 + 1) + b(a^2P_1 + 1) \le 1$.

When X_1, X_2 are independent we have

$$\begin{aligned} &I(X_1; T_1|X_2, T_2, S_1) - I(X_1; Y_2|X_2, T_2, S_1) \\ &= I(X_1; X_1 + \eta_1\dot{Z}_1) - I(X_1; aX_1 + \sqrt{1-\rho_2^2}\ddot{Z}_2) \end{aligned}$$

which is concave in $p_1(x_1)$ when $a^2\eta_1^2 \le 1 - \rho_2^2$ (by stochastic degradation of the virtual receivers). Thus if $a^2\eta_1^2 \le 1 - \rho_2^2$ we have that

$$\mathcal{G}[I(X_1; T_1|X_2, T_2, S_1, q) - I(X_1; Y_2|X_2, T_2, S_1, q)] = 0.$$

A similar statement holds when $b^2\eta_2^2 \le 1 - \rho_1^2$.

Hence the outer bound reduces to

$$SR \le \max_{\substack{Q, X_1, X_2 \\ E(X_1^2) \le P_1, \mathrm{E}(X_2^2) \le P_2}} \Big\{ I(X_1; T_1, Y_1|Q) + I(X_2; T_2, Y_2|Q) \Big\}.$$

It is a rather routine exercise (using EPI by expanding into differential entropies and consider suitable differences) to verify that under this choice of parameters the optimal choice of inputs for X_1 and X_2 are Gaussians and under this distribution $X_k \to Y_k \to T_k$ forms a Markov chain for $k = 1, 2$ Hence

$$SR \le C\left(\frac{P_1}{b^2P_2 + 1}\right) + C\left(\frac{P_2}{a^2P_1 + 1}\right).$$

4 Conclusion

This paper defines an enhanced version of the genie-based sum-rate outer bound. We prove that for continues channels with independent Gaussian noises, the enhanced genie-based sum-rate outer bound is tight for all cases where sum-rate capacity is previously known. Additional advantages of genie-based outer bounds will be explored in the future. An immediate question is whether this reduces the 1-bit gap to capacity established in [3].

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