Non-convex Optimization and Network Information Theory

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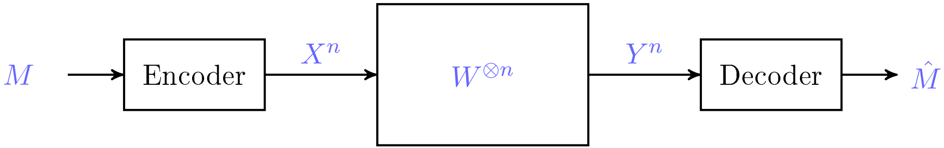
Non-convex problems and network information theory

- Introduction
  - ⋆ Building blocks
  - ⋆ How to test the optimality of coding schemes
  - ⋆ Where do the optimization problems arise?

- Two problems to illustrate some ideas

- Observations and potential future directions

Point-to-point communication



A rate R is achievable if there exists a sequence of encoding/decoding maps so that $\mathbf{P}(M \neq \hat{M}) \to 0$ as $n \to \infty$. Capacity, $C(W) := \sup\{R : R$ is achievable $\}$.

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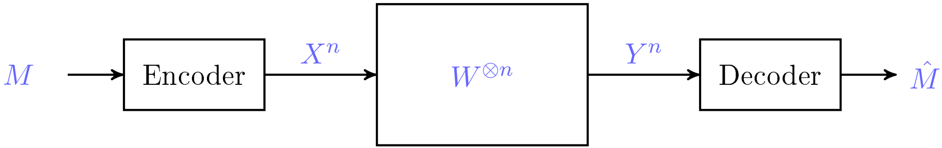

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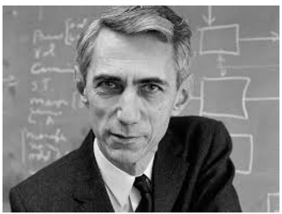

Shannon

Random coding can be used to achieve

$$R(W) = \sup_{\mu(x)} I(X;Y)$$

where $\quad I(X;Y) := \sum_{x,y} \mu_{X,Y}(x,y) \log\left(\frac{\mu_{X,Y}(x,y)}{\mu_X(x)\mu_Y(y)}\right)$

$I(X;Y)$: mutual information between X and Y

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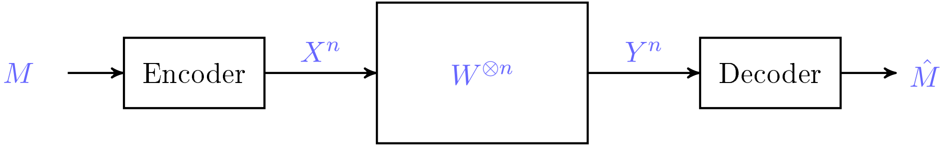

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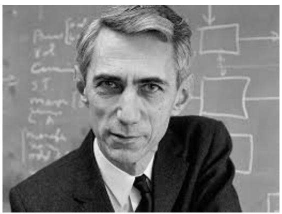

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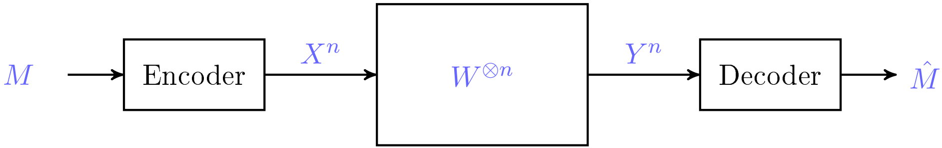

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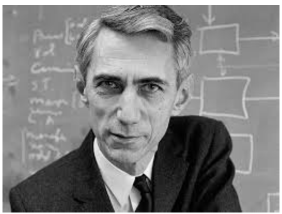

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$I(X;Y)$: mutual information between X and Y

Question: Is $R(W) = C(W)$? **(YES)** (Shannon '48)

Testing optimality

It is easy (why?) to see that $R(W)$ is optimal `if and only if`

$$R(W \otimes W) = 2R(W) \quad \forall\, W.$$

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If $\exists W$ such that $\frac{1}{2}R(W \otimes W) > R(W)$ then

$$C(W) \geq \frac{1}{2}R(W \otimes W) > R(W)$$

(Hence equality is necessary)

Testing optimality

It is easy (why?) to see that $R(W)$ is optimal **`if and only if`**

$$R(W \otimes W) = 2R(W) \quad \forall\, W.$$

Given $\epsilon > 0$ there is a sequence of codes such that

$$\frac{1}{n} I(X^n; Y^n) \geq C(W) - \epsilon, \quad \forall n > n_0$$

- Fano's inequality
- Data processing inequality

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Hence, for k such that $N = 2^k > n_0$ we have

$$R(W) \overset{\text{indc}}{=} \frac{1}{N} R(\underbrace{W \otimes \cdots \otimes W}_{N}) = \frac{1}{N} I(X^N; Y^N) \geq C(W) - \epsilon.$$

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$$I(X_1, X_2; Y_1, Y_2) \leq I(X_1; Y_1) + I(X_2; Y_2).$$

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Sub-additivity

A functional defined over a probability simplex is said to be sub-additive if

$$F_{12}(\mu_{X_1,X_2}) \leq F_1(\mu_{X_1}) + F_2(\mu_{X_2}) \quad \forall\, \mu_{X_1,X_2}.$$

In above, since W is fixed, $I(X;Y)$ is a functional over μ_X, the space of input distributions.

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$$\begin{aligned}
I(X_1, X_2; Y_1, Y_2) &= I(X_1, X_2; Y_1) + I(X_1, X_2; Y_2|Y_1) \\
&= I(X_1, X_2; Y_1) + I(Y_1, X_1, X_2; Y_2) - I(Y_1; Y_2) \\
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$$\begin{aligned} I(X_1, X_2; Y_1, Y_2) &= I(X_1, X_2; Y_1) + I(X_1, X_2; Y_2 | Y_1) \\ &= I(X_1, X_2; Y_1) + I(Y_1, X_1, X_2; Y_2) - I(Y_1; Y_2) \\ &= I(X_1; Y_1) + I(X_2; Y_2) - I(Y_1; Y_2) \\ &\leq I(X_1; Y_1) + I(X_2; Y_2). \end{aligned}$$

Note: Computing $R(W) = \sup_{p(x)} I(X;Y)$ is relatively easy, since $I(X;Y)$ is a concave function of $p(x)$.

Successes

The various ideas introduced by Shannon have led to an information revolution

Random coding and its optimality have directly inspired

- Low density parity check codes (LDPC)
- Polar codes
  - ⋆ proof of sub-additivity

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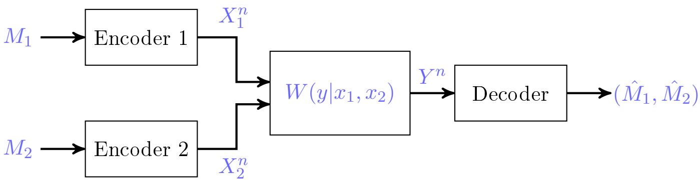
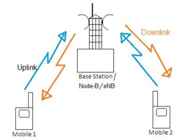
We are now (fully immersed) in a wireless world

- Network of users sharing same medium
- Clear need to maximally utilize the limited resources (power, bandwidth, energy)
- Develop a similar understanding in network settings
  - ⋆ But we first need to fully understand the basic building blocks

1. Multiple Access Channel (uplink) (Shannon '61)





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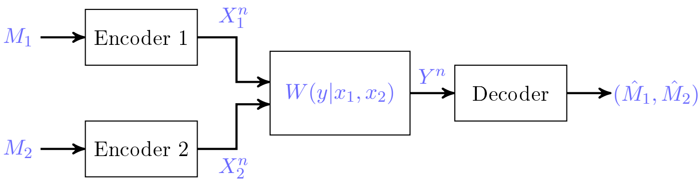

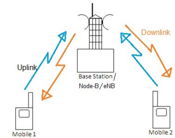

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Ahlswede

Random coding can be used to achieve rate pairs (R_1, R_2) that satisfy

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\begin{aligned}
R_1 &\leq I(X_1; Y | X_2, Q) \\
R_2 &\leq I(X_2; Y | X_1, Q) \\
R_1 + R_2 &\leq I(X_1, X_2; Y | Q)
\end{aligned}
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for some $p(q)p(x_1|q)p(x_2|q)$; it suffices to consider $|\mathcal{Q}| \leq 2$. Call this region $\mathcal{R}(W)$.

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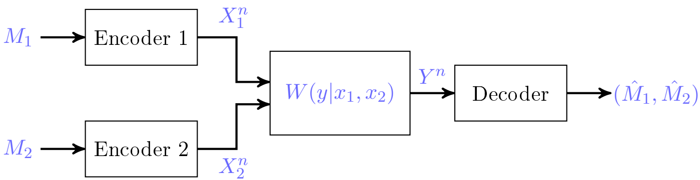

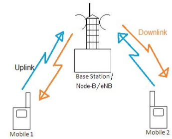

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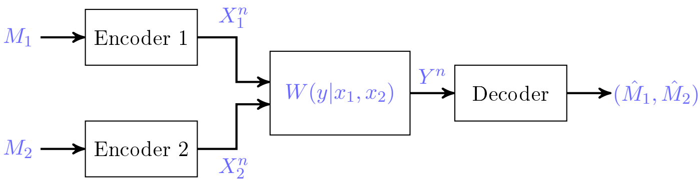

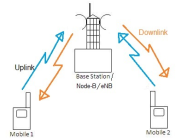

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Question: Is this the capacity (optimal) region? **(YES)** (Ahlswede '72)

Testing optimality

Define, for $\lambda \geq 1$,

$$\begin{aligned} S_\lambda(W) &= \max_{(R_1,R_2)\in\mathcal{R}(W)} \left\{\lambda R_1 + R_2\right\} \\ &= \max_{p_1(x_1)p_2(x_2)} \Big\{(\lambda - 1)I(X_1;Y|X_2) + I(X_1,X_2;Y)\Big\} \end{aligned}$$

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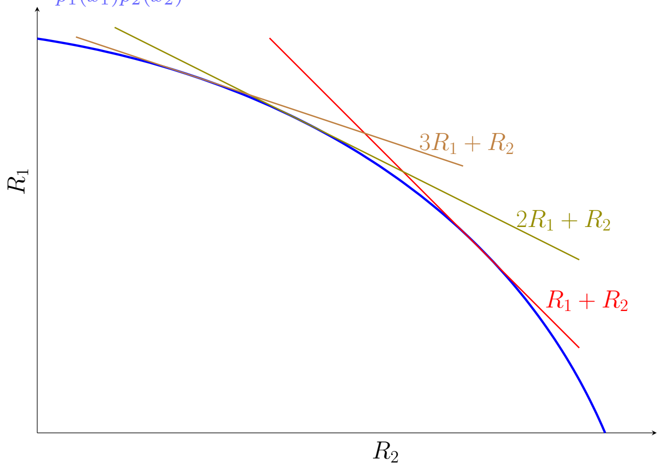

Supporting hyperplanes

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As before, $\mathcal{R}(W)$ is optimal `if and only if`

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&\quad\leq (\lambda - 1)I(X_{11};Y_1|X_{21}) + I(X_{11},X_{21};Y_1)\\
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One can establish this in same way as point-to-point setting.

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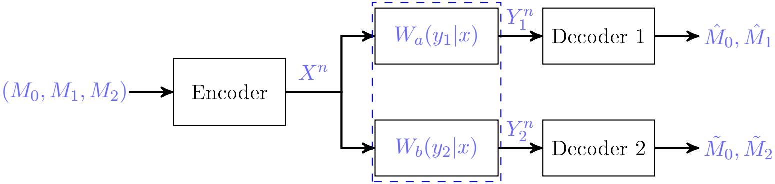
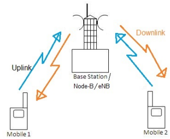
Note: Computing $S_\lambda(W)$ is relatively easy since $\left\{(\lambda-1)I(X_1;Y|X_2) + I(X_1,X_2;Y)\right\}$ is concave in $p_1(x_1), p_2(x_2)$.

2. Broadcast channel (downlink) (Cover '72)





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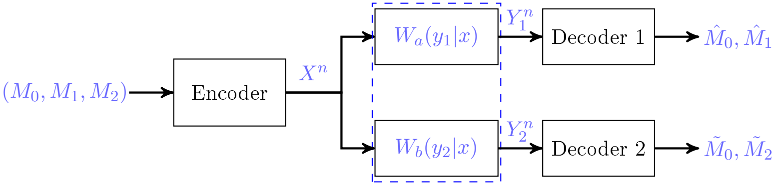

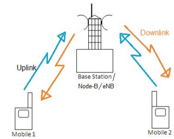

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Marton

Superposition coding and random hashing can be used to achieve rate triples (R_0, R_1, R_2) that satisfy

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R_0 &\leq \min\{I(Q;Y_1), I(Q;Y_2)\} \\
R_0 + R_1 &\leq I(U,Q;Y_1) \\
R_0 + R_2 &\leq I(V,Q;Y_2) \\
R_0 + R_1 + R_2 &\leq \min\{I(Q;Y_1), I(Q;Y_2)\} + I(U;Y_1|Q) \\
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for some $p(q,u,v,x)$. Call this region $\mathcal{R}(W_a, W_b)$.

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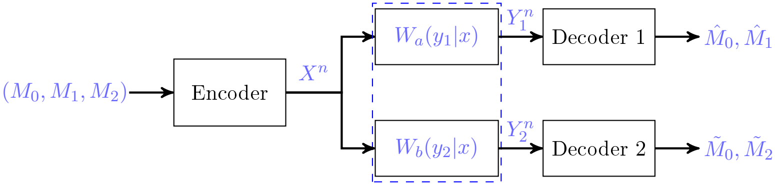

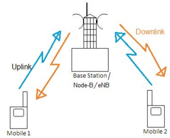

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Question: Is this the capacity (optimal) region?

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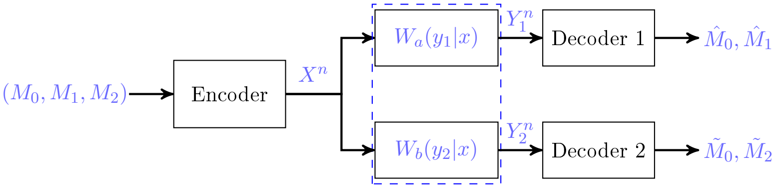

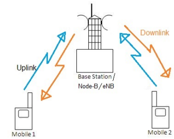

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for some $p(q,u,v,x)$. Call this region $\mathcal{R}(W_a, W_b)$.

Question: Is this the capacity (optimal) region? **(Open)** (since Marton '79)

Testing optimality ($R_0 = 0$)

Define, for $\lambda \geq 1$,

$$\begin{aligned}
S_\lambda(W) &= \max_{(R_1,R_2)\in\mathcal{R}(W_a,W_b)} \{\lambda R_1 + R_2\} \\
&= \max_{p(u,v,w,x)} \Big\{(\lambda-1)I(U,Q;Y_1) + \min\{I(Q;Y_1), I(Q;Y_2)\} + I(U;Y_1|Q) \\
&\qquad\qquad + I(V;Y_2|Q) - I(U;V|Q)\Big\} \\
&= \min_{\alpha\in[0,1]} \max_{p(u,v,w,x)} \Big\{(\lambda-\alpha)I(Q;Y_1) + \alpha I(Q;Y_2) + \lambda I(U;Y_1|Q) \\
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\end{aligned}$$

As before, $\mathcal{R}(W_a, W_b)$ is optimal `if and only if`

$$S_\lambda(W_a \otimes W_a, W_b \otimes W_b) = 2S_\lambda(W_a, W_b) \quad \forall\, W_a, W_b, \lambda \geq 1.$$

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&\qquad\qquad + I(V;Y_2|Q) - I(U;V|Q)\Big\} \\
&= \min_{\alpha\in[0,1]} \max_{p(u,v,w,x)} \Big\{(\lambda-\alpha)I(Q;Y_1) + \alpha I(Q;Y_2) + \lambda I(U;Y_1|Q) \\
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\end{aligned}
$$

As before, $\mathcal{R}(W_a, W_b)$ is optimal **if and only if**

$$S_\lambda(W_a \otimes W_a, W_b \otimes W_b) = 2S_\lambda(W_a, W_b) \quad \forall\, W_a, W_b, \lambda \geq 1.$$

Note: Computing $S_\lambda(W_a, W_b)$ is a non-convex optimization problem.

Successes

In spite of the underlying problem being intrinsically non-convex

- $\mathcal{R}(W_a, W_b)$ is optimal on $R_1 = 0$ (or $R_2 = 0$)
  - ⋆ Degraded message sets: Korner and Marton ('77)
- $\mathcal{R}(W_a, W_b)$ is optimal for some classes of channels
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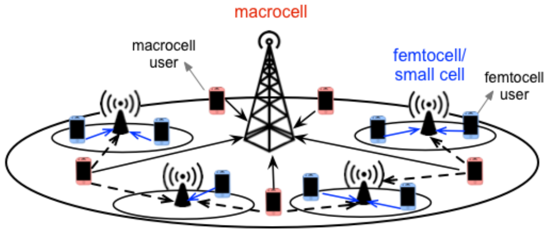
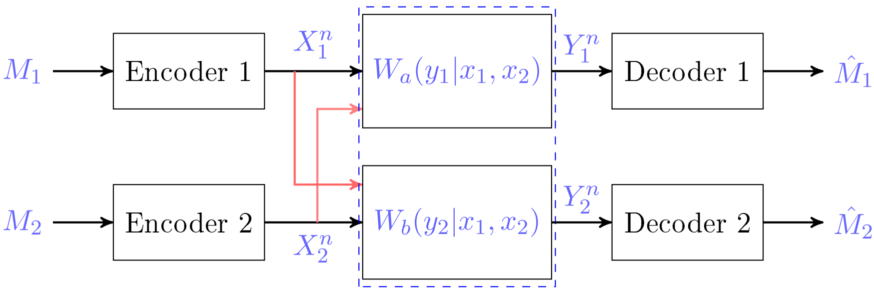
  - ⋆ **Geng-Nair** '14: Optimality of $\mathcal{R}(W_a, W_b)$ for Gaussian broadcast channel: Technique for establishing extremality of Gaussian distributions using sub-additivity of functionals

3. Interference Channel (Ahlswede '74)



Credit: *www.personal.psu.edu/bxg215/research.html*



3. Interference Channel (Ahlswede '74)

Han

Kobayashi

Superposition coding, message splitting, coded time-sharing can be used to achieve rate pairs (R_1, R_2) that satisfy

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for some pmf $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$, where $|\mathcal{U}_1| \le |\mathcal{X}_1| + 4$, $|\mathcal{U}_2| \le |\mathcal{X}_2| + 4$, and $|\mathcal{Q}| \le 7$. Call this region $\mathcal{R}(W_a, W_b)$.

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Question: Is this the capacity region?

Had been **open** (since Han and Kobayashi '81)

Successes

In spite of the underlying problem being intrinsically non-convex

- $\mathcal{R}(W_a, W_b)$ is optimal for some classes of channels
  - ⋆ Carleial '75, Sato '81, El Gamal and Costa ('81 and '86)
- $\mathcal{R}(W_a, W_b)$ is close to optimal for Gaussian Interference channel
  - ⋆ Etkin and Tse and Wang ('09)
- Novel ideas and mathematical results came out from the investigation of optimality
  - ⋆ Concavity of entropy power (Costa '85)
  - ⋆ Genie based approach to prove sub-additivity (El Gamal and Costa '81, Kramer '02)

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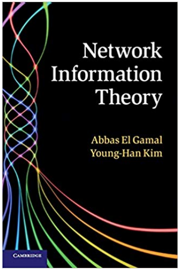
  - ⋆ Concavity of entropy power (Costa '85)
  - ⋆ Genie based approach to prove sub-additivity (El Gamal and Costa '81, Kramer '02)
- $\mathcal{R}(W_a, W_b)$ is not optimal in general (Nair, Xia, Yazdanpanah '15)

Broadcast and interference channels are far too important

- To let non-convexity dissuade us
- To not investigate simple classes that require new ideas

A class of open problems

A sub-collection of the 15 open problems listed in Chaps. 5-9.



5.1 `What is the capacity region of less noisy broadcast-channels with four or more receivers?`
(two-receiver: Korner-Marton '76, three-receiver: Nair-Wang '10)

5.2 `What is the capacity region of more capable broadcast-channels with three or more receivers?`
(two-receiver: El Gamal '79)

6.1 `What is the capacity region of the Gaussian Interference channel with weak interference?`
(strong-interference: Sato '79; mixed-interference corner-points: Sato' 81, Costa'85; weak-interference corner-points: rate-sum (partial): three-groups '09)

6.4 `Is the Han-Kobayashi inner bound tight in general for interference channels?`

8.2 `Is superposition coding optimal for the general 3-receiver broadcast channel with one message to all three receivers and another message to two receivers?`

8.3 `What is the sum-capacity of the binary skew-symmetric broadcast channel?`

8.4 `Is the Marton inner bound tight in general for broadcast channels?`

9.2 `Can the converse for the Gaussian broadcast channel be proved directly by optimizing the Nair-El Gamal outer bound?`

9.3 `What is the capacity region of the 2-receiver Gaussian broadcast channel with common message?`

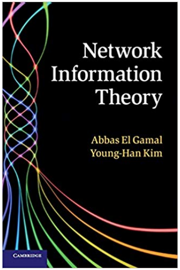
A class of open problems

My reformulations of a few of them.

5.1 Is superposition coding optimal for less-noisy broadcast channels with four or more receivers?

5.2 Is superposition coding optimal for more-capable broadcast channels with three or more receivers?

6.1 Is the Han-Kobayashi scheme with Gaussian signaling tight for the Gaussian Interference channel with weak interference?



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The common theme to these (reformulated) questions

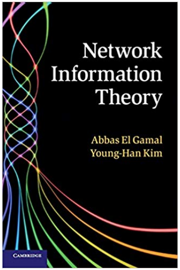
Common theme

Is a candidate rate region optimal?

Idea for testing optimality:

- $S_\lambda(W \otimes W) \stackrel{?}{=} 2S_\lambda(W)$
- Determine sub-additivity of an associated non-convex functional

Status of the open problems



5.1 Is superposition coding optimal for less-noisy broadcast channels with four or more receivers?(**OPEN**)

5.2 Is superposition coding optimal for more-capable broadcast channels with three or more receivers? (**NO**: **Nair-Xia** '12)

6.1 Is the Han-Kobayashi scheme with Gaussian signaling tight for the Gaussian Interference channel with weak interference?(OPEN) (**YES**: **corner points** using ideas in measure transportation by **Polyanskiy-Wu** '15)

6.4 Is the Han-Kobayashi inner bound tight in general for interference channels? (**NO**: **Nair-Xia-Yazdanpanah** '15)

8.2 Is superposition coding optimal for the general 3-receiver broadcast channel with one message to all three receivers and another message to two receivers? (**NO**: **Nair-Yazdanpanah** '17)

8.3 Does the Marton inner bound achieve the sum-capacity of the binary skew-symmetric broadcast channel?(**OPEN**)

8.4 Is the Marton inner bound tight in general for broadcast channels?(**OPEN**)

9.2 Can the converse for the Gaussian broadcast channel be proved directly by optimizing the Nair-El Gamal outer bound?(**YES**: **Geng-Nair** '14)

9.3 Does the Marton inner bound achieve the capacity region of the 2-receiver Gaussian broadcast channel with common message?(**YES**: **Geng-Nair** '14)

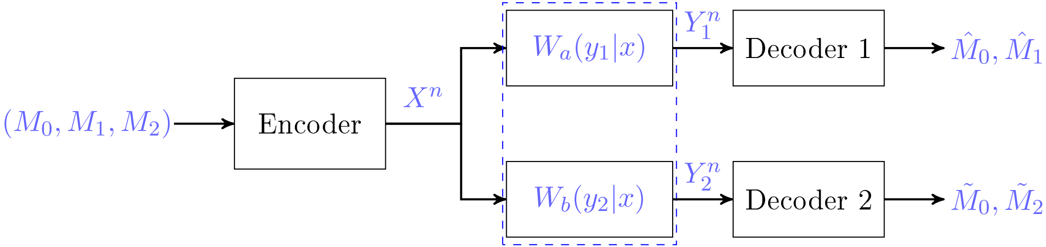
Outline

- Broadcast channel: Establishing optimality of Marton's region for MIMO broadcast channel

- Interference channel: Sub-optimality of the Han–Kobayashi region

- Family of non-convex optimization problems
  - ⋆ Relation to problems of interest in other fields
  - ⋆ Unifying observations and some conjectures

MIMO (Vector) Gaussian broadcast channel



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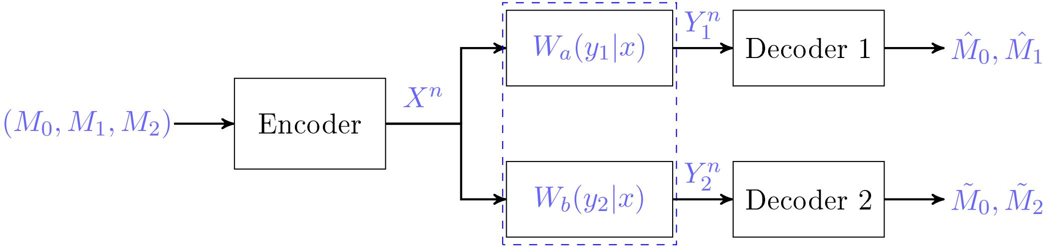

MIMO Gaussian broadcast channel:

$$Y_1 = AX + Z$$
$$Y_2 = BX + Z$$

where $Z \sim \mathcal{N}(0, I)$ denotes the additive noise.

Very important channel class in wireless communication

Models: multi-antenna transmitter/receivers (downlink)

History

Optimality of Marton's bound, $\mathcal{R}(W_a, W_b)$, was established:

- Scalar case (Bergmans '73) (Entropy Power Inequality)
- Reversely degraded setting (Poltyrev '78, El Gamal '81)
- Optimality on $R_0 = 0$ (Weingarten and Steinberg and Shamai '06)
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Explain our technique on $R_0 = 0$ (for simplicity)

Outer bound (Korner-Marton '79)

The set of rate pairs (R_1, R_2) satisfying

$$R_2 \le I(U;Y_2)$$
$$R_1 + R_2 \le I(U;Y_2) + I(X;Y_1|U)$$

for some $p(u,x)$, where $\mathrm{E}(\|X\|^2) \le P$ forms an outer bound to the capacity region.

Denote this region as $\mathcal{O}(W_a, W_b)$.

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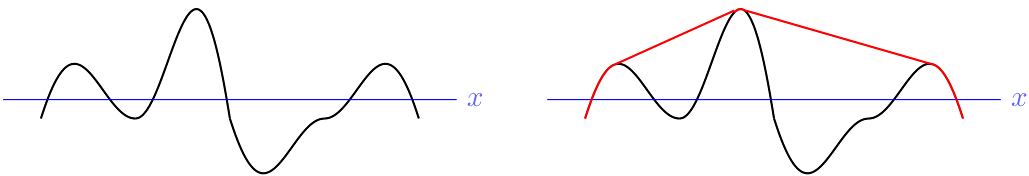

$f(x)$ upper concave envelope: $\mathcal{C}_x[f]$

One can show that if Gaussians maximize

$$\mathcal{C}_{\mu_X}[h(Y_1) - \lambda h(Y_2)]$$

then Marton's inner bound is optimal (on $R_0 = 0$)

Here $h(X)$ is the differential entropy:

$$h(X) := -\int f(x) \log f(x) dx,$$

where $f(x)$ is the density function of X.

A similar (more-involved) problem shows up when $R_0 \neq 0$

An identical technique (to the one I am going to demonstrate) establishes that also

Gaussian optimality via sub-additivity (Geng-Nair '14)

Maximize, for $\lambda > 1$, the value of the functional

$$\mathcal{C}_{\mu_X}[h(AX + Z) - \lambda h(BX + Z)]$$

over $X : \mathrm{E}(XX^T) \preceq K$, where A, B are invertible matrices and $Z \sim \mathcal{N}(0, I)$.

We will see that the maximum value is

$$h(AX_* + Z) - \lambda h(BX_* + Z),$$

where $X_* \sim \mathcal{N}(0, K')$ for some $K' \preceq K$.

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Lemma: $\mathcal{C}_{\mu_X}[h(AX+Z) - \lambda h(BX+Z)]$ is sub-additive.

Proof: For any μ_{X_1,X_2}

$$\begin{aligned}
&h(AX_1+Z_1, AX_2+Z_2|U) - \lambda h(BX_1+Z_1, BX_2+Z_2|U) \\
&= h(AX_1+Z_1|U, AX_2+Z_2) - \lambda h(BX_1+Z_1|U, AX_2+Z_2) \\
&\quad + h(AX_2+Z_2|U, BX_1+Z_1) - \lambda h(BX_2+Z_2|U, BX_1+Z_1) \\
&\quad\quad - (\lambda-1) I(AX_2+Z_2; BX_1+Z_1|U)
\end{aligned}$$

Gaussian optimality via sub-additivity (Geng-Nair '14)

Maximize, for $\lambda > 1$, the value of the functional

$$\mathcal{C}_{\mu_X}[h(AX+Z) - \lambda h(BX+Z)]$$

over $X : \mathrm{E}(XX^T) \preceq K$, where A, B are invertible matrices and $Z \sim \mathcal{N}(0, I)$.

We will see that the maximum value is

$$h(AX_* + Z) - \lambda h(BX_* + Z),$$

where $X_* \sim \mathcal{N}(0, K')$ for some $K' \preceq K$.

Lemma: $\mathcal{C}_{\mu_X}[h(AX+Z) - \lambda h(BX+Z)]$ is sub-additive.

Proof: For any μ_{X_1,X_2}

$$\begin{aligned}
&\mathcal{C}_{\mu_{X_1,X_2}}[h(AX_1+Z_1, AX_2+Z_2) - \lambda h(BX_1+Z_1, BX_2+Z_2)] \\
&\le \mathcal{C}_{\mu_{X_1}}[h(AX_1+Z_1) - \lambda h(BX_1+Z_1)] \\
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&= h(AX_1+Z_1|U, AX_2+Z_2) - \lambda h(BX_1+Z_1|U, AX_2+Z_2)\\
&\quad + h(AX_2+Z_2|U, BX_1+Z_1) - \lambda h(BX_2+Z_2|U, BX_1+Z_1)\\
&\quad\quad -(\lambda-1)I(AX_2+Z_2; BX_1+Z_1|U)
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Gaussian optimality: ctd..

Let $(U_\dagger, X_\dagger)$ be a maximizer, i.e.

$$V = \max_{\mu_X} \mathcal{C}_{\mu_X}[h(AX+Z) - \lambda h(BX+Z)] = h(AX_\dagger + Z|U_\dagger) - \lambda h(BX_\dagger + Z|U_\dagger).$$

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Setting $U = (U_a, U_b)$, $X_+ = \frac{X_a + X_b}{\sqrt{2}}$ and $X_- = \frac{X_a - X_b}{\sqrt{2}}$ the proof of sub-additivity yields

$$\begin{aligned}
2V &= \mathcal{C}_{\mu_{X_1,X_2}}[h(AX_1+Z_1, AX_2+Z_2) - \lambda h(BX_1+Z_1, BX_2+Z_2)]\Big|_{(\mu_{X_+,X_-})} \\
&\le \mathcal{C}_{\mu_{X_1}}[h(AX_1+Z_1) - \lambda h(BX_1+Z_1)]\Big|_{\mu_{X_+}} \\
&\quad + \mathcal{C}_{\mu_{X_2}}[h(AX_2+Z_2) - \lambda h(BX_2+Z_2)]\Big|_{\mu_{X_-}} \\
&\quad\quad -(\lambda-1) I(AX_- + Z_2; BX_+ + Z_1 | U_a, U_b) \\
&\le V + V
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&\le V + V
\end{aligned}$$

Therefore: we get that conditioned on (U_a, U_b): $X_+ \perp X_-$.

Gaussian optimality: ctd..

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Let (X_a, U_a) and (X_b, U_b) be i.i.d. according to $(U_\dagger, X_\dagger)$.

Note: Thus, conditioned on (U_a, U_b):

- $X_a \perp X_b$ (from construction)
- $(X_a + X_b) \perp (X_a - X_b)$ (from proof of sub-additivity)

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  - ⋆ Characterization of Gaussians (Bernstein '40s)
  - ⋆ Proof: Using characteristic functions (Fourier transforms)

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This technique has been subsequently used by others in various other instances.

Note: There are some similarities with work of Lieb and Barthe (90s)

They also use rotations (but not information measures and its algebra)

An open question

We have seen (yesterday and today) how sub-additivity implies Gaussian optimality

Open question

For $\alpha, a \in (0,1)$, do Gaussians maximize the functional

$$\alpha h(X_2 + aX_1 + Z) + (1-\alpha)h(X_1 + Z) - h(aX_1 + Z)$$

over $X_1 \perp X_2$, subject to $\mathrm{E}(X_1^2) \leq P_1$, $\mathrm{E}(X_2^2) \leq P_2$. Here $Z \sim \mathcal{N}(0,1)$ is independent of X_1, X_2.

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Affirmative if the following functional sub-additive?

$$\mathcal{C}_{\mu_{X_1}} \left[\alpha h(X_2 + aX_1 + Z) + (1-\alpha) h(X_1 + Z) - h(aX_1 + Z) \right]$$

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Why should someone care?

- If true, solves the capacity region for the Gaussian Z-interference channel
- Related to reverse EPIs, hyperplane conjecture, etc.

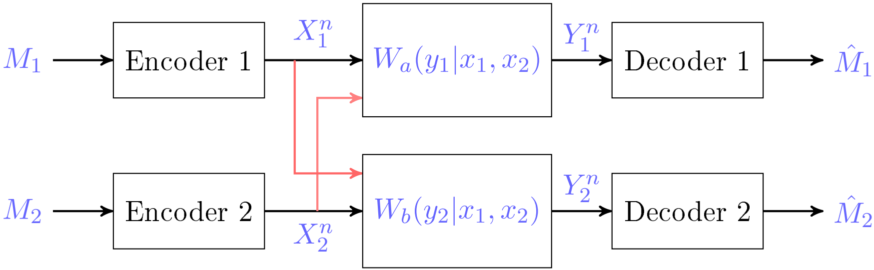
Outline

- Broadcast channel: Establishing optimality of Marton's for MIMO broadcast channel

- Interference channel: Sub-optimality of the Han–Kobayashi region

- Family of non-convex optimization problems
  - ⋆ Relation to problems of interest in other fields
  - ⋆ Unifying observations and some conjectures

Interference Channel (Ahlswede '74)



Han-Kobayashi achievable region (1981) á la Chong et. al.

A rate-pair (R_1, R_2) is achievable for the interference channel if

$$
\begin{aligned}
R_1 &< I(X_1; Y_1 | U_2, Q), \\
R_2 &< I(X_2; Y_2 | U_1, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q), \\
R_1 + R_2 &< I(X_2, U_1; Y_2 | Q) + I(X_1; Y_1 | U_1, U_2, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1 | U_1, Q) + I(X_2, U_1; Y_2 | U_2, Q), \\
2R_1 + R_2 &< I(X_1, U_2; Y_1 | Q) + I(X_1; Y_1 | U_1, U_2, Q) + I(X_2, U_1; Y_2 | U_2, Q), \\
R_1 + 2R_2 &< I(X_2, U_1; Y_2 | Q) + I(X_2; Y_2 | U_1, U_2, Q) + I(X_1, U_2; Y_1 | U_1, Q)
\end{aligned}
$$

for some pmf $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$, where $|U_1| \leq |X_1| + 4$, $|U_2| \leq |X_2| + 4$, and $|Q| \leq 7$. Denote the (closure of) region as $\mathcal{R}(W_a, W_b)$.

Numerically infeasible to compute $\mathcal{R}(W_a W_b)$ even for generic binary-input binary-output interference channels

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R_1 + R_2 &< I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q),\\
2R_1 + R_2 &< I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q),\\
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\end{aligned}$$

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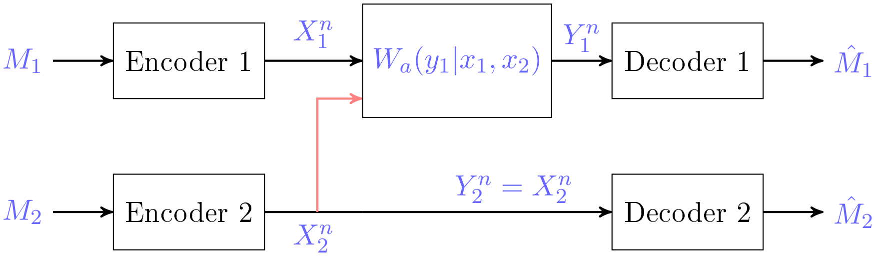
Numerically infeasible to compute $\mathcal{R}(W_a W_b)$ even for generic binary-input binary-output interference channels

First step:

- Find a channel class where HK region simplifies **AND** yet not too trivial

Clean Z Interference Channel (CZIC) Model



Clean Z-interference channel

Lemma: A rate-pair (R_1, R_2) belongs to Han-Kobayashi region if and only if

$$\begin{aligned} R_1 &< I(X_1; Y_1|U_2, Q), \\ R_2 &< H(X_2|Q), \\ R_1 + R_2 &< I(X_1, U_2; Y_1|Q) + H(X_2|U_2, Q), \end{aligned}$$

for some pmf $p(q)p(x_1|q)p(u_2, x_2|q)$, where $|U_2| \leq |X_2|$ and $|Q| \leq 2$.

Denote region: $\mathcal{R}(W_a)$

Testing optimality

Equivalent to test if

$$S_\lambda(W_a \otimes W_a) = 2S_\lambda(W_a),\ \forall\ W_a, \lambda \geq 0,$$

where

$$S_\lambda(W_a) := \max_{(R_1,R_2)\in\mathcal{R}(W_a)} \lambda R_1 + R_2.$$

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For $\lambda \in [0,1]$, $S_\lambda(W_a)$ is given by

$$\begin{aligned}
&\max_{p_1(x_1)p_2(u_2,x_2)} \Big\{(1-\lambda)H(X_2) + \lambda I(X_1,U_2;Y_1) + \lambda H(X_2|U_2)\Big\} \\
&= \max_{p_1(x_1)p_2(x_2)} \Big\{H(X_2) + \lambda I(X_1;Y_1)\Big\}
\end{aligned}$$

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&= \max_{p_1(x_1)p_2(x_2)} \Big\{H(X_2) + \lambda I(X_1; Y_1)\Big\}
\end{aligned}$$

Lemma (sub-additivity) (Nair-Xia-Yazdanpanah '15):

$$\begin{aligned}
&H(X_{21}, X_{22}) + \lambda I(X_{11}, X_{12}; Y_{11}, Y_{12}) \\
&\quad \leq \Big\{H(X_{21}) + \lambda I(X_{11}; Y_{11})\Big\} + \Big\{H(X_{22}) + \lambda I(X_{12}; Y_{12})\Big\} - (1-\lambda)I(X_{21}; X_{22}).
\end{aligned}$$

Implies optimality of $S_\lambda(W_a)$, $\lambda \in [0,1]$.

What about $\lambda > 1$?

For $\lambda \geq 1$, $S_\lambda(W_a)$ is given by

$$\begin{aligned}
&\max_{p_1(x_1)p_2(u_2,x_2)} \Big\{ I(X_1, U_2; Y_1) + H(X_2|U_2) + (\lambda - 1) I(X_1; Y_1|U_2) \Big\} \\
&= \max_{p_1(x_1)p_2(x_2)} \Big\{ I(X_1, X_2; Y_1) + \mathcal{C}_{X_2}[(\lambda - 1) I(X_1; Y_1) + H(X_2) - I(X_2; Y_1|X_1)] \Big\}
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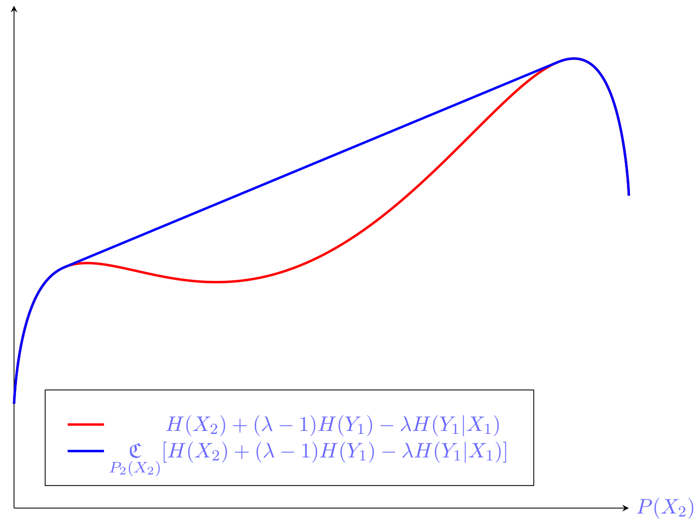
\end{aligned}$$

Question: Can we numerically test if $S_\lambda(W_a \otimes W_a) = 2S_\lambda(W_a)$?

X_2 is a binary random variable (i.e. concave envelope over single variable)
$(\lambda - 1) I(X_1; Y_1) + H(X_2) - I(X_2; Y_1|X_1)$: has at most 2 inflexion points

What about $\lambda > 1$?



The shape of concave envelope for a generic binary CZIC

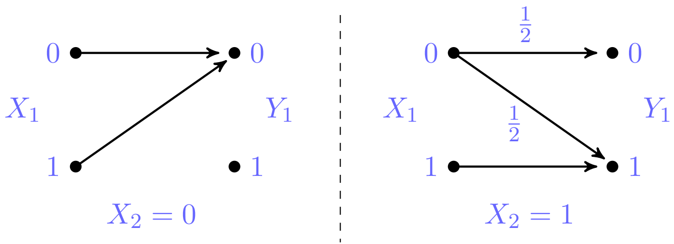
Sub-optimality of the Han-Kobayashi region

λ	$W(Y_1 = 0 \vert X_1, X_2)$	$\mathscr{A}_\lambda^{\mathrm{HK}}(W)$	$\frac{1}{2}\mathscr{A}_\lambda^{\mathrm{TIN}}(W^{\otimes 2})$
2	$\begin{bmatrix} 1 & 0.5 \\ 1 & 0 \end{bmatrix}$	1.107516	1.108141
9	$\begin{bmatrix} 0.12 & 0.89 \\ 0.21 & 0.62 \end{bmatrix}$	1.074484	1.075544
12	$\begin{bmatrix} 0.01 & 0.58 \\ 0.20 & 0.74 \end{bmatrix}$	1.289830	1.293760
14	$\begin{bmatrix} 0.78 & 0.07 \\ 0.46 & 0.05 \end{bmatrix}$	1.426526	1.432419
15	$\begin{bmatrix} 0.91 & 0.22 \\ 0.66 & 0.15 \end{bmatrix}$	1.323766	1.339065
16	$\begin{bmatrix} 0.91 & 0.13 \\ 0.62 & 0.06 \end{bmatrix}$	1.515421	1.534724
18	$\begin{bmatrix} 0.38 & 0.87 \\ 0.12 & 0.79 \end{bmatrix}$	1.449959	1.468577

Counterexamples to the optimality of Han-Kobayashi region.

Note: For the first example, we can calculate the concave envelope analytically.

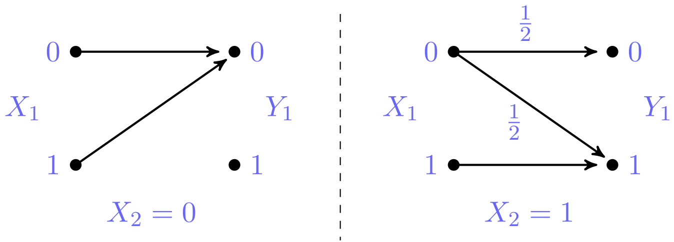
Particular Channel



- We compute $\max\limits_{HK} \; \lambda R_1 + R_2$ for $\lambda = 2$

$$\max_{p_1(x_1)p_2(x_2)} \Big(H(Y_1) + \mathop{\mathfrak{C}}_{p_2(x_2)} \big[H(X_2) + 2H(Y_1) - H(Y_1|X_1) \big] \Big)$$

Particular Channel



- We compute $\max\limits_{HK} \; \lambda R_1 + R_2$ for $\lambda = 2$

$$\max_{p_1(x_1)p_2(x_2)} \Big(H(Y_1) + \underset{p_2(x_2)}{\mathfrak{C}} \big[\overbrace{H(X_2) + 2H(Y_1) - H(Y_1|X_1)}^{f(p,q)}\big] \Big)$$

- Let p and q respectively denote $Pr(X_1 = 0)$ and $Pr(X_2 = 0)$

$$f(p,q) = (1 - 2\bar{p})h_b(q) + h_b(q + \frac{p}{2}\bar{q}) - 2ph_b(\frac{q+1}{2})$$

where $h_b(.)$ is the binary entropy function

Particular channel continued

$f(p,q)$ is concave in q for $p \geq \frac{1}{2}$ and for $0 \leq p < \frac{1}{2}$

$$\underset{q}{\mathfrak{C}}[f(p,q)] = \begin{cases} f(p,q) & q > 1-2p \\ \dfrac{f(p,1-2p) - f(p,0)}{1-2p} q + f(p,0) & q \in [0, 1-2p] \end{cases}$$

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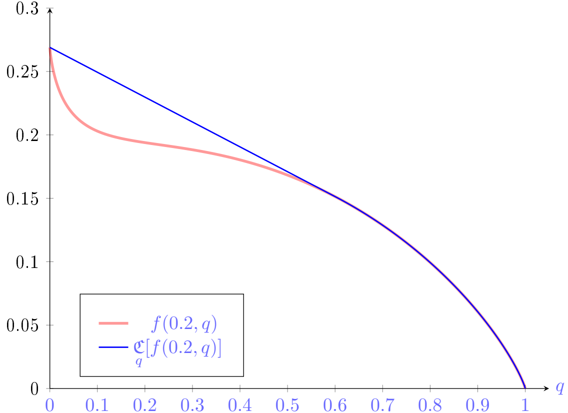

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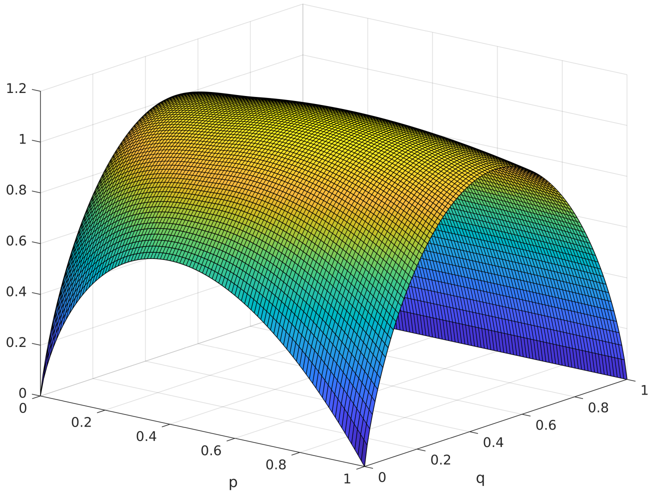
Corollary

Maximum of $2R_1 + R_2$ for the Han–Kobayashi region is equal to the maximum of $T(p,q)$ for $(p,q) \in [0,1] \times [0,1]$, where

$$T(p,q) = \begin{cases} h_b(q + \frac{p}{2}\bar{q}) + f(p,q) & q \geq \min\{0, 1-2p\} \\ h_b(q + \frac{p}{2}\bar{q}) + \dfrac{f(p,1-2p) - f(p,0)}{1-2p} q + f(p,0) & o.w., \end{cases}$$

where $f(p,q) = (1-2\bar{p})h_b(q) + h_b(q + \frac{p}{2}\bar{q}) - 2p h_b(\frac{q+1}{2})$

Plot of $T(p,q)$



Numerical search indicates: $\max_{p,q} T(p,q) = 1.107516..$ at $p = 0.5078..$ and $q = 0.4365..$

Particular channel continued

- Interval arithmetic is a method to obtain formal bounds for functions consisting of basic arithmetic functions and commonly used functions such as logarithms and trigonometric functions.
- $T(p, q)$ only includes basic arithmetic functions and logarithm.
- We used Julia based implementation of this formal method to obtain

$$\max T(p, q) \in [1.10751, 1.10769]$$

- The 2-letter TIN achieves $2R_1 + R_2 = 1.108141$ at the distribution

$$P((X_{11}, X_{12}) = (0, 0)) = p \qquad P((X_{11}, X_{12}) = (1, 1)) = 1 - p$$
$$P((X_{21}, X_{22}) = (0, 0)) = 0.36q \qquad P((X_{21}, X_{22}) = (1, 1)) = 1 - 1.64q$$
$$P((X_{21}, X_{22}) = (0, 1)) = 0.64q \qquad P((X_{21}, X_{22}) = (1, 0)) = 0.64q$$

  where $p = 0.507829413$, $q = 0.436538150$
- Repetition coding across time seems to outperform memoryless coding

What about Marton's region for the broadcast channel?

Is the following functional sub-additive or is there an example where it is super-additive?

Let $W_a(y|x)$ and $W_b(z|x)$ be given channels, $\alpha \in [0,1]$, and $\lambda \geq 1$.

$$\mathcal{C}_{\mu_X}\left[(\lambda-\alpha)H(Y)-\alpha H(Z)+\max_{p(u,v|x)}\{\lambda I(U;Y)+I(V;Z)-I(U;V)\}\right]$$

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Remarks:

- Conjectured to be sub-additive (Anantharam-Gohari-Nair '13)
- To evaluate the concave envelope
  - ⋆ Suffices to consider (U,V): $|U|+|V| \leq |X|+1$.
  - ⋆ We did not get any contradiction to sub-addivity for binary input broadcast channels
- Can prove sub-additivity when $\alpha = 0$ or $\alpha = 1$.

Remarks

- Idea: To demonstrate **super-additivity**
- **Difficulty**: Identify a sufficiently simple class where
  - ⋆ Evaluation of the region is possible : non-convex optimization
  - ⋆ Super-additivity holds

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- Idea: To demonstrate **super-additivity**
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  - ⋆ Evaluation of the region is possible : non-convex optimization
  - ⋆ Super-additivity holds

This idea was also used to resolve

8.2 Is superposition coding optimal for the general 3-receiver DM-BC with one message to all three receivers and another message to two receivers?
NO (Nair,Yazdanpanah '17)

Outline

- Broadcast channel: Establishing optimality of Marton's for MIMO broadcast channel

- Interference channel: Sub-optimality of the Han–Kobayashi region

- Family of non-convex optimization problems
  - ⋆ Relation to problems of interest in other fields
  - ⋆ Unifying observations and some conjectures

A specific family of non-convex optimization problems

Shows up: Testing the optimality of coding schemes

Testing optimality (usually) reduces to testing sub-additivity of

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Using Fenchel duality this is same as

$$G_1(\gamma_1) := \max_{\mu_{\mathbf{X}}} \sum_{S\subseteq[n]} \alpha_S H(X_S) - \mathrm{E}(\gamma_1(\mathbf{X}))$$

$$G_2(\gamma_2) := \max_{\mu_{\mathbf{X}}} \sum_{S\subseteq[n]} \alpha_S H(X_S) - \mathrm{E}(\gamma_2(\mathbf{X}))$$

$$G_{12}(\gamma_1,\gamma_2) := \max_{\mu_{\mathbf{X}_1,\mathbf{X}_2}} \sum_{S\subseteq[n]} \alpha_S H(X_{1S}, X_{2S}) - \mathrm{E}(\gamma_1(\mathbf{X}_1)) - \mathrm{E}(\gamma_2(\mathbf{X}_2))$$

Is $G_{12}(\gamma_1,\gamma_2) = G_1(\gamma_1) + G_2(\gamma_2) \ \forall \ \gamma_1, \gamma_2$?
i.e. Is the maximizer of G_{12} a product distribution?

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Are there other fields where the same family shows up?

Hypercontractivity

Studied in functional analysis, cs theory, etc.

Definition

$(X,Y) \sim \mu_{XY}$ is (p,q)-hypercontractive for $1 \leq q \leq p$ if

$$\|Tg\|_p \leq \|g\|_q \quad \forall g(Y)$$

where T is the Markov operator characterized by $\mu_{Y|X}$

Here $\|Z\|_p = E(|Z|^p)^{\frac{1}{p}}$.

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There is a lot of interest in evaluting hypercontractivity parameters for distributions.

Theorem (Nair '14)

$(X, Y) \sim \mu_{XY}$ is (p, q)-hypercontractive for $1 \leq q \leq p$ if and only if

$$\mathcal{C}_{\nu_{X,Y}}\left[H(X,Y) - (1 - \frac{1}{p})H(X) - \frac{1}{q}H(Y)\right]\Big|_{\mu_{X,Y}} = H(X,Y) - (1 - \frac{1}{p})H(X) - \frac{1}{q}H(Y)$$

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Hypercontractivity parameters satisfies a property called tensorization:

If $(X_1, Y_1) \perp (X_2, Y_2)$ are both (p, q)-hypercontractive, then $((X_1, X_2), (Y_1, Y_2))$ is also (p, q)-hypercontractive

Gets around the curse of dimensionality.

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Rather immediate that sub-additivity, i.e.

$$\begin{aligned}
&\mathcal{C}_{\mu_{X_1Y_1X_2Y_2}}[H(X_1Y_1X_2Y_2) - \lambda_1 H(X_1X_2) - \lambda_2 H(Y_1Y_2)] \\
&\quad \leq \mathcal{C}_{\mu_{X_1Y_1}}[H(X_1Y_1) - \lambda_1 H(X_1) - \lambda_2 H(Y_1)] + \mathcal{C}_{\mu_{X_2Y_2}}[H(X_2Y_2) - \lambda_1 H(X_2) - \lambda_2 H(Y_2)]
\end{aligned}$$

is equivalent to tensorization of hypercontractivity parameters

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This (serendipitous) rediscovery of the link between hypercontractivity and information measures and these equivalent characterizations is spurring a lot of work

Consequences

Computation of hypercontractivity parameters is considered hard

- X is uniform and $\mu_{Y|X}$ is binary symmetric channel
  - ⋆ (Bonami-Beckner inequality '70s, Borrell '82)
- (X, Y) Jointly Gaussian (Gross '75)

Evaluation of achievable regions is of similar difficulty as determining hypercontractivity (same family and similar terms)

For testing optimality of schemes we had to develop tools for evaluating achievable regions for certain channels

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Yes, we can.

E.g.: X is uniform and $\mu_{Y|X}$ is binary erasure channel (Nair-Wang '16,'17)

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Other techniques we used to solve these non-convex problems:

- Identify a lower dimensional manifold that contains all the stationary points
- Analyze the function directly on this manifold or
- Use properties of the points on this manifold to deduce sub-additivity

Recap

Test the optimality of coding schemes in network information theory

- Resolved some open questions
- Many remain open

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Computed the optimizers of several non-convex functionals

- Developed some new tools and techniques
    - ⋆ Gaussian optimality via sub-additivity
    - ⋆ Optimal auxiliaries correspond to computation of concave envelopes
    - ⋆ Min-max theorem
- More ideas and tools seem necessary

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Test the optimality of coding schemes in network information theory

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These (specific family) non-convex functionals also appear in other fields

- The tools (already) developed can be used to get some new results

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An Observation

Reminder: Family of functionals that showed up in network information theory

$$\sum_{S\subseteq[n]} \alpha_S H(X_S), \ \ \alpha_S \in \mathbb{R}.$$

Usually, one is interested in testing the sub-additivity of

$$\mathcal{C}_{\mu_X}[\alpha_S H(X_S)].$$

This is equivalent to testing a global tensorization property.

Definition

A functional $\sum_{S\subseteq[n]} \alpha_S H(X_S)$ is said to satisfy **global tensorization** if a **`product distribution maximizes`** $G^{\mu}_{12}(\gamma_1, \gamma_2)$ for all γ_1, γ_2, where

$$G^{\mu}_{12}(\gamma_1, \gamma_2) := \sum_{S\subseteq[n]} \alpha_S H(X_{1S}, X_{2S}) - \mathrm{E}(\gamma_1(\mathbf{X}_1)) - \mathrm{E}(\gamma_2(\mathbf{X}_2))$$

An Observation

Definition

A functional $\sum_{S\subseteq[n]} \alpha_S H(X_S)$ is said to satisfy **local tensorization** if `the product of local maximizers` of $G^{\mu_1}(\gamma_1)$ and $G^{\mu_2}(\gamma_2)$ `is a local maximizer` of $G^{\mu}_{12}(\gamma_1, \gamma_2)$ for all γ_1, γ_2, where

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Observation (Conjecture)

For functionals in this family global tensorization holds if and only if local tensorization holds

Note: Similarity to testing concavity using a local (second derivative) condition

Notes

For some of the remaining open problems (mentioned earlier), we can establish local-tensorization

- Marton's inner bound for binary input broadcast channels
- Gaussian Z-interference channel

Therefore, if the Conjecture is true, then we would establish the capacity region for these settings

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Therefore, if the Conjecture is true, then we would establish the capacity region for these settings

Question: How may these two phenomena be connected?

A possible answer is (again) suggested by our computations in various examples

Conjecture 2

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Consider

$$f_\alpha(\gamma) = \max_{\mu_{\mathbf{X}}} \sum_{S \subseteq [n]} \alpha_S H(X_S) - \mathrm{E}(\gamma(\mathbf{X})), \quad \alpha_S \in \mathbb{R}.$$

Suppose $\alpha_S^{(0)}$ and $\alpha_S^{(1)}$ have interior global maximizers.
Let $\alpha_S^{(t)} = (1-t)\alpha_S^{(0)} + t\alpha_S^{(1)}$, $t \in [0,1]$. Then there exists a continuous path in the simplex such that $\mu^{(t)}$ is a global maximizer of $f_{\alpha^{(t)}}(\gamma)$ for all $t \in [0,1]$.

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Consequences:

- Information theory: Conjecture 2 (plus mild regularity conditions) implies the Conjecture that local tensorization implies global tensorization
- Algorithms: Suppose one wants to approximate hypercontractivity parameters
  - ⋆ Start with $p \to \infty$
  - ⋆ Approximate the maximizing distribution at this boundary value of norm.
  - ⋆ Decrease p and track the global maximizers by local search.

Optimization based approaches

Optimization based approaches have been game changers

First jump: Linear programming to convex optimization

Semi-definite program based algorithm design and analysis

- Compressive sensing
- Phase recovery
- Clustering
- Image processing

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New Jump: Convex optimization to specific families of non-convex optimization

Studies on these families are already making impact in

- Machine learning and AI (Singular Value Decomposition)
- Graphical models and Statistical Physics based approaches (sum of energy and entropy terms)
- Communication networks (linear combination of entropies of subsets)

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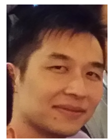

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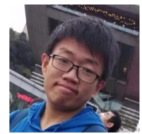
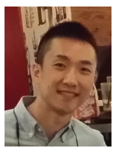
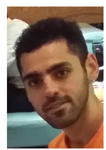


Yan Nan Wang



Lingxiao Xia



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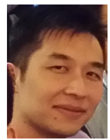
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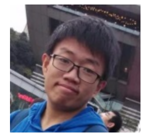
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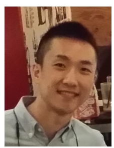
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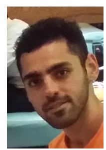
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THANK YOU