Proportional Fairness in Multi-channel Multi-rate Wireless Networks – Part I: Theory, Algorithms, and Application to Capacity Allocation Problem in Large-Scale WiFi Networks

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Abstract – This is Part I of a two-part paper series that studies the use of the proportional fairness (PF) utility function as the basis for capacity allocation and scheduling in multi-channel multi-rate wireless networks. The contributions of Part I are threefold. (i) First, we lay down the theoretical foundation for PF. Specifically, we present the fundamental properties and physical/economic interpretation of PF. We show by general mathematical arguments that PF leads to equal airtime allocation to users for the single-channel case; and equal *equivalent airtime* allocation to users for the multi-channel case, where the equivalent airtime enjoyed by a user is a weighted sum of the airtimes enjoyed by the user on all channels, with the weight of a channel being the *price* or *value* of that channel. We also establish the Pareto efficiency of PF solutions. (ii) Second, we derive characteristics of PF solutions that are useful for the construction of PF-optimization algorithms. We present several PF-optimization algorithms, including a fast algorithm that is amenable to parallel implementation. (iii) Third, we study the use of PF utility for capacity allocation in large-scale WiFi networks consisting of many adjacent wireless LANs. We find that the PF solution simultaneously achieves higher system throughput, better fairness, and lower outage probability with respect to the default solution given by today's 802.11 commercial products.

Key words– **Proportional fairness, Scheduling, AP association, Capacity assignment, WLAN, 802.11, Wireless networks**

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I. INTRODUCTION

Capacity allocation is a fundamental problem in communications networks when there are competing demands from users for the network bandwidth. When allocating capacity to network users, there is generally a tradeoff between system throughput and fairness. Fig. 1 (a) and (b) are simple illustrating examples. Fig. 1(a) shows a wired network with three nodes: *a*, *b*, and *c*. A link connects nodes *a* and *b*, and another link connects nodes *b* and *c*. The capacity of each link is one unit. There are three traffic flows: flow 1 from node *a* to node *b*; flow 2 from node *b* to node *c*; and flow 3 from node *a* to node *c*. Let T_i be the throughput of flow *i*. To maximize the system throughput $\sum_i T_i$, we should set $T_1 = T_2 = 1$ and $T_3 = 0$, for a total throughput of 2. This solution, however, is totally unfair to flow 3. On the other hand, if the goal is to be fair so that we maximize $\min_i (T_i)$, then $T_1 = T_2 = T_3 = 1/2$. The total throughput will then be $3/2 < 2$.

Fig. 1(b) shows a wireless network with two wireless stations. Because of their different distances to the base station, the transmission rate of link 1 is b_1 bps, and that of link 2 is b_2 bps, with $b_1 >> b_2$. At any one time, only one of the links can be in use because they share the same wireless medium. Let P_i be the fraction of airtime used by link *i*. Then, the throughput of station *i* is $T_i = P_i b_i$ and the system throughput is $P_1 b_1 + P_2 b_2$. If the goal is to maximize system throughput, then $P_1 = 1$ and $P_2 = 0$. If we maximize $\min_i (T_i)$ for fairness, then $P_1 = b_2/(b_1 + b_2)$ and $P_2 = b_1/(b_1 + b_2)$. Note that if $b_2 \approx 0$, this yields system throughput ≈ 0 (i.e., trying to achieve equal throughputs among users may cause the system throughput to be dragged down by a poor performing station [1]).

Since the publication of Ref. [2], there has been growing interest in capacity allocation based on maximizing the log utility function, $y = \sum_i \log(T_i)$. With this utility function, the solution to Fig. 1(a) becomes $T_1 = T_2 = 2/3$, $T_3 = 1/3$; and the solution to Fig. 1(b) becomes $P_1 = P_2 = 1/2$, yielding $T_1 = b_1/2$, $T_2 = b_2/2$. The log utility is also referred to as the proportional fairness (PF) utility. Most prior work, particularly those related to *wired networks*, adopts the PF utility so as to strike a balance between system throughput and fairness. However, the implicit assumption that user happiness, or the perceived quality of service, increases according to the log of its throughput has not been established on a solid foundation. There also appears to be no direct physical justification for the use of PF utility.

In contrast, the use of PF utility in *wireless networks* has an appealing physical justification. We show in this paper that in multi-rate wireless networks, maximizing PF utility is equivalent to allocating equal *airtime* to all users, hence establishing a physical correspondence to the use of PF utility. That is, maximizing PF in the throughput domain is equivalent to max min fairness in the airtime domain. If we treat airtime as the previous resource for which the users compete in a wireless network, then maximizing PF yields fair resource allocation.

Earlier, a paper co-authored by one of us [3] made this observation of IEEE 802.11 networks. The current paper, however, shows that this result is generic and applies to all multi-rate wireless networks (i.e., not just 802.11 networks). The result is also valid for wireless networks with multiple channels except that in place of *airtime*, we need the concept of *equivalent airtime*, in which the airtime of a particular channel is weighted the "value" or "price" of the channel. Maximizing PF will then correspond to allocating equal equivalent airtime to all users.

Problem Formulation and Motivating Examples

The formulation in this paper is quite general and applies to capacity allocation and scheduling problems in various settings. Two scenarios are shown in Fig. 2 and Fig. 3. Fig. 2 concerns the problem of assigning wireless stations to wireless local area networks (WLAN). There are U wireless stations (STA) and S wireless access points (AP) distributed over a geographical region. Suppose that adjacent APs operate on different frequency channels (e.g., in 802.11a, there are eight available orthogonal frequency channels) so that there is no co-channel interference among the WLANs. Then, essentially we have S channels in the system. The data transmission rate enjoyed by STA i if it connects to channel k (AP k) is $b_{i,k}$, where $b_{i,k}$ is a function of the signal-to-noise ratio (SNR) with respect to channel k (e.g., in 802.11a, there are eight possible data transmission rates: 6, 9, 12, 18, 24, 36, 48, or 54Mbps, so that $b_{i,k}$ is one of these values, or 0Mbps if STA i is too far away from AP k).

In current 802.11 networks, an STA usually associates itself with the AP with the strongest signal. This may lead to load imbalance and uneven throughputs among STAs when the distribution of STAs is not uniform. To avoid this problem, we could solve the optimization problem as follows:

$$\begin{aligned} & \max y = \sum_{i=1}^{U} \log\left(\sum_{k=1}^{S} P_{i,k} b_{i,k} \right) \\ & s.t. \ \sum_{i=1}^{U} P_{i,k} = 1 \ \ \forall k = 1, \cdots, S \\ & \quad P_{i.k} \geq 0 \quad \forall i = 1, \cdots, U; k = 1, \cdots, S \end{aligned} \tag{1}$$

where $P_{i,k}$ is the fraction of airtime of channel k used by STA i, and $y = \sum_i \log\left(\sum_k P_{i,k} b_{i,k}\right)$ is the PF utility function. Note that we have assumed the integration of the capacity assignment and scheduling problems in the above formulation. In particular, we assume that once airtimes $P_{i,k}$ are determined, there is a medium access control (MAC) scheduling protocol that will make sure that user i uses no more that $P_{i,k}$ fraction of the airtime of channel k when all STAs are busy. The reader is referred to [3] for such MAC protocols.

Fig. 3 depicts an OFDM system [4] in which the air channel is divided into S subcarriers which can be dynamically assigned to users. For the downlink (uplink), the base station can simultaneously transmit to (receive from) multiple users on different subcarriers. With respect to Fig. 3, $P_{i,k}$ in (1) corresponds to the fraction of the airtime of subcarrier k allocated to user i, and $b_{i,k}$ is the bit rate of user i on subcarrier k.

This paper series considers the problem formulation as in (1) and its variations. Besides deriving the theories and general results that apply to different multi-channel multi-rate settings, we focus on the application scenario of Fig. 2 to show how to apply the PF algorithms in system designs. The remainder of this paper is organized as follows. Section II presents general results related to interpretations of PF optimality in multi-channel wireless networks. Section III gives several characteristics of PF optimal solutions that help the construction of PF algorithms and interpretation of numerical results later. Section IV presents several PF algorithms. Section V makes use of one of the algorithms to generate numerical results for the application scenario of Fig. 2. Section V concludes Part I of the paper series.

II. INTERPRETATIONS OF PF OPTIMALITY IN MULTI-CHANNEL WIRELESS NETWORKS

This section presents two fundamental properties associated with the *multi-channel multi-rate PF optimization problem* in (1). These properties give us an economic interpretation for PF optimality. Specifically, we show the following:

1. In the PF-optimal solution, the users are allocated equal *equivalent airtime*.
2. The PF-optimal solution is Pareto efficient.

A. Single-channel Case and Conditions for PF-Optimality in the Multi-channel Case.

Ref. [3] considered scheduling in 802.11 WLAN to achieve PF optimality, which corresponds to the single channel case (S = 1) under our general setting here. The main result is that PF optimality in the throughput domain is equivalent to max-min fairness in the airtime domain, and that the 802.11 MAC protocol could be configured to achieve PF optimality easily. As shown below, the essence of this conclusion is in principle true for all wireless networks (including cellular networks), and not just 802.11 networks. To focus on the fundamental, we shall ignore protocol overhead in the following discussion. Interested readers are referred to [3] for how such overhead can be taken into account in 802.11 networks.

Consider a wireless network with S = 1, and let us label the sole channel as channel 1. At any one time, only one user can transmit. We assume that there is enough traffic in the network so that it is always busy. Pick a random point in time. Let $P_{i,1}$ be the probability of finding user i transmitting. The throughput of user i is $T_i = P_{i,1} b_{i,1}$. It can be easily shown that the optimal solution to (1) is obtained by setting $P_{i,1} = 1/U$ for all i.

The above formulation is quite general. For a time-slotted system with time slots of fixed duration, PF optimality means that each user is equally likely to transmit in a given time slot. For a packet system with fixed packet size (in byte), the packet duration (in second) of user i is proportional to $1/b_{i,1}$. To satisfy the optimal condition $P_{i,1} = 1/U$, the underlying scheduling scheme should make sure that an arbitrary transmitted packet is that from user i with probability $b_{i,1} / \sum_i b_{i,1}$. This interpretation can be easily mapped into 802.11 MAC either by varying the contention window (CW) or transmission opportunity (TXOP) among the users [3]. In general, the system does not even have to adopt probabilistic scheduling. As long as the system schedules user i to transmit $P_{i,1} = 1/U$ fraction of the time, it is PF optimal.

PF optimality in the single-channel case has a nice and simple interpretation: users should have equal shares of airtime. This makes economic sense in situations where the users are subscribers who pay the same subscription fee to the service provider. In [1], it was shown that a user that transmits at very low rate because of poor SNR can easily drag down the performance of all other users in an 802.11 WLAN, because of the excessive airtime it uses. As a result, everybody suffers because of the "poor" user. With PF scheduling, this problem can be removed, because equal airtime usage establishes a sort of "firewall" among users [3].

One can of course generalize the concept to situations where different users have different priorities (see Extension to Theorem 1 in Section 2.B) and should therefore be allocated different amounts of airtimes. The key concept, however, is that "airtime" is the resource that should be meted out to the users carefully rather than raw throughput.

We shall see that unlike in a single-channel system, PF optimality in a multi-channel system does not mean equal "physical" airtime usage. The airtime of each channel must first be weighted by a "shadow price". Once that is done the concept of an equivalent airtime can then be defined so that PF optimality means equal equivalent airtime among all users. We first present the Karush-Kuhn-Tucker (KKT) conditions for the optimization problem of (1) below.

Karush-Kuhn-Tucker (KKT) Conditions for Multi-channel PF Optimality

We now turn our attention to the multi-channel problem in (1). Let $[P_{i,k}]$ be the matrix representing a feasible solution, in which rows correspond to users, and columns correspond to channels. Thanks to the concavity of y in the feasible region, the following KKT conditions [5] are necessary and sufficient for a feasible solution $[P^*_{i,k}]$ (with corresponding $T^*_i = \sum_k P^*_{i,k} b_{i,k}$) to be optimal:

1. For each channel k, for each pair of users i and j with $P^*_{i,k} > 0$ and $P^*_{j,k} = 0$,

$$\left.\frac{\partial y}{\partial P_{i,k}}\right|_{P^*_{i,k}} \geq \left.\frac{\partial y}{\partial P_{j,k}}\right|_{P^*_{j,k}} . \tag{2}$$

That is, $$b_{i,k} \big/ T^*_i \geq b_{j,k} \big/ T^*_j . \tag{3}$$

2. For each channel k, for each pair of users i and j with $P^*_{i,k} > 0$ and $P^*_{j,k} > 0$,

$$\left.\frac{\partial y}{\partial P_{i,k}}\right|_{P^*_{i,k}} = \left.\frac{\partial y}{\partial P_{j,k}}\right|_{P^*_{j,k}} . \tag{4}$$

That is, $$b_{i,k} \big/ T^*_i = b_{j,k} \big/ T^*_j . \tag{5}$$

2-User-2-Channel Example

Let $[b_{i,k}]$ be the matrix consisting of the bit rates of different users on different channels. Consider a 2-user-2-channel example in which

$$[b_{i,k}] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

It can be verified that the solution

$$[P^*_{i,k}] = \begin{bmatrix} 1 & 1/4 \\ 0 & 3/4 \end{bmatrix}$$

satisfies the KKT conditions and is therefore optimal.

We observe the following about the optimal solution: 1) the two users do not have equal airtime on each channel: in fact, user 2 has zero airtime on channel 1; 2) neither are the sums of airtimes on the channels equal: user 1 has total airtime of 1.25, and user 2 has total airtime of 0.75, on the two channels. So, the equal-airtime property of PF optimality in the single channel case does not carry over to the multi-channel case directly.

B. *Equivalent Airtime in Multi-channel Problem*

In the above 2-user-2-channel example, if we weight the airtime on each channel by its "shadow price",

$$\left.\frac{\partial y}{\partial P_{i,k}}\right|_{P_{i,k}^*} = \frac{b_{i,k}}{T_i^*} \tag{6}$$

where $P_{i,k}^* > 0$, then the total weighted airtime of user 1 is $b_{1,1}/T_1^* + 0.25 b_{1,2}/T_1^* = 1$; and the total weighted airtimes of user 2 is $0.75 b_{2,2}/T_2^* = 1$. So, the total weighted airtimes of the two users are equal.

The interpretation is as follows. In the above example, both users can transmit at higher bit rates on channel 2. So, channel 2 is *more valuable* than channel 1. The shadow price of a channel *k* is a measure of the "value" of the airtime of the channel. Specifically, it is a measure of the potential increase in the utility function *y* for each unit increase in airtime on channel *k* under optimality (i.e., how much *y* could be increased further if the constraint $\sum_i P_{i,k} = 1$ could be relaxed). Of course, physically, the sum of the fractions of airtimes used by all users on channel *k* cannot be increased to beyond 1. Nevertheless, this does not invalidate the use of the shadow price as a "pricing mechanism" for our optimization problem.

We now formally show that the equivalent airtime usage of all users must be equal for multi-channel PF optimality. Consider an optimal solution $[P_{i,k}^*]$. Let λ_k^* be the shadow price of channel *k*. Define the equivalent airtime usage of user *i* as $E_i = \sum_k \lambda_k^* P_{i,k}^* = \sum_{k\in K_i^*} \lambda_k^* P_{i,k}^*$ where K_i^* is the subset of channels in which $P_{i,k}^* > 0$.

Theorem 1: $E_i = 1 \quad \forall i$

Proof: $E_i = \sum_{k\in K_i^*} \lambda_k^* P_{i,k}^* = \left(\sum_{k\in K_i^*} b_{i,k} P_{i,k}^*\right) / T_i^* = 1$ □

Extension of Theorem 1: Suppose that user *i* is willing to pay a subscription cost of c_i, and we modify the utility function in (1) to $y = \sum_i c_i \log\left(\sum_k P_{i,k} b_{i,k}\right)$. Then $E_i = c_i \quad \forall i$.

Proof: In this case, the shadow price is

$$\lambda_k^* = c_i b_{i,k} / T_k^* .$$

So, $$E_i = \sum_{k\in K_i^*} \lambda_k^* P_{i,k}^* = c_i \left(\sum_{k\in K_i^*} b_{i,k} P_{i,k}^* \right) \Big/ T_i^* = c_i \quad (7)$$

□

We see from the above extension that users may get varying amounts of equivalent airtimes according to the costs they pay.

C. Pareto Efficiency of PF Optimality

The PF utility function is just one of many possible utility functions that can serve as the optimization criterion. Within the feasible region defined by the constraints in (1), there are many feasible solutions, each corresponding to one achievable set of throughputs among the users. Generally, there is a tradeoff among the throughputs enjoyed by different users so that increasing the throughput of one user means decreasing the throughput of another user. When such a tradeoff exists, one cannot say for sure whether one feasible solution is better than the other: much depends on the utility function being adopted. However, some of the solutions do have *ranking* among them so that we can establish that one solution is superior to the other regardless of the utility assumed. This requires the concept of Pareto efficiency borrowed from the field of economics.

Let $T=[T_i]$ be the vector representing the user throughputs in a feasible solution. We say that $[T_i']\succ[T_i]$ if $T_i'\geq T_i$ for all *i*, and there is at least an *i* such that $T_i'>T_i$. So, a solution yielding $[T_i']$ is superior to another solution yielding $[T_i]$ in the sense that no user has lower throughput in the former than in the latter, but there is at least one user with higher throughput in the former. Note that the concept of ranking is independent of the utility function adopted. If a solution is ranked higher than another solution, no matter what utility function is used, it is still superior to the other solution.

Definition of Pareto Efficiency: A feasible solution yielding $T=[T_i]$ is Pareto efficient if and only if one cannot identify another feasible solution yielding $T'=[T_i']$ such that $T'\succ T$.

The optimal solutions under utilities of PF and max system throughput are both Pareto-efficient. The Pareto efficiency of the PF solution can be proven quite trivially, as in Theorem 2. We assume that there is no user *i* such that $b_{i,k}=0$ for all *k*; otherwise, user *i* should be removed from consideration since $T_i=0$ regardless of the optimization process, and the PF utility function will always be negative infinity.

Definition of Strong PF in Multi-channel Optimization: We define "strong" PF to mean that all users *i* with $b_{i,k}=0$ for all *k* should be removed from consideration in the optimization process. PF optimality in this paper means strong PF optimality.

Theorem 2: A PF-optimal solution is Pareto-efficient.

Proof: Suppose the optimal solution yields the utility $y^* = \sum_i \log T_i^*$ and that the solution is not Pareto efficient. Then we can find another solution that yields $y' = \sum_i \log T_i'$ such that $T_i' \geq T_i^*$ for all i, and there is at least an i such that $T_i' > T_i^*$. In other words, $y' > y^*$, and therefore y^* could not have been the optimal solution under PF. □

III. Characteristics of PF Optimal Solutions

This section discusses characteristics of PF optimal solution useful for the construction of optimization algorithms. We will also use these characteristics to interpret the numerical results in Section V.

A. Numbers of Shared and Exclusively Assigned Channels

In the 2-user-2-channel example in Section 2, we see that in the optimal solution, one channel is shared and one channel is exclusively assigned to one user. At the same time, one user uses just one channel while the other user uses both channels. It turns out that in a *U*-user-*S*-channel system, there is an optimal solution in which there are at most $U-1$ shared channels, and at most $S-1$ users using more than one channel.

Definition of Shared Channels:

1. A channel *k* is said to be shared if there are at least two non-zero P_{ik}, $i \in I_U$, where I_U is the set of all users in the system.
2. A channel *k* is shared among *N* ($N \leq U$) users in the system if there are at least two non-zero P_{ik}, $i \in I_N$, where $i \in I_N$, and $I_N \subseteq I_U$ is the subset containing the *N* users.
3. N ($N \leq U$) users are said to share *K* ($K \leq S$) channels if each and every of the *K* channels is shared among the *N* users.

Theorem 3: Consider a system with *U* users and *S* channels. There is an optimal solution in which the number of shared channels among any *N* of the *U* users is no more than $N-1$.

Corollary 1: For a *U*-user-*S*-channel system, there is an optimal solution with no more than $\min(S, U-1)$ shared channels, and with at least $\max(0, S-U+1)$ channels that are exclusively used by just one user.

Proof of Corollary 1: Obvious from Theorem 3.

*Proof of Theorem 3***:** Consider an optimal solution which yields throughput T_i^* for user *i*. Suppose that in this solution, there are *N* users sharing $K \geq N$ channels (see Definition 3 above on shared channels). We show that we can find another optimal solution such that the *N* users share no more than $N-1$ channels.

Consider the representation of a feasible solution by a bipartite graph in which the left vertices represent the users and the right vertices represent the channels, and in which there is an edge between a vertex *i* on the left and a vertex *k* on the right if $P_{i,k} > 0$. For the bipartite graph to be loop-free, there can be no more than $(N+K-1)$ edges. The bipartite graph corresponding to the original optimal solution above contains a loop, because according to our definition of shared channels, there are at least $2K \geq N+K$ edges in the original solution.

Loop-removal Procedure

We present a procedure that shifts probability assignments to remove loops while maintaining the throughput T_i^* for each user *i*. Consider a loop with *n* left vertices and *n* right vertices. Label the left vertices i_1, i_2, .., i_n; and the right vertices k_1, k_2, …, k_n, with the edges in the loop being (i_1, k_1), (i_2, k_1), (i_2, k_2), (i_3, k_2), …, (i_n, k_n), (i_1, k_n). The KKT conditions for optimality require

$$\begin{aligned} T_{i_1}^* / T_{i_2}^* &= b_{i_1 k_1} / b_{i_2 k_1} \\ T_{i_2}^* / T_{i_3}^* &= b_{i_2 k_2} / b_{i_3 k_2} \\ &\vdots \\ T_{i_{n-1}}^* / T_{i_n}^* &= b_{i_{n-1} k_{n-1}} / b_{i_n k_{n-1}} \end{aligned} \tag{8}$$

and $$T_{i_n}^* / T_{i_1}^* = b_{i_n k_n} / b_{i_1 k_n} = (b_{i_2 k_1} / b_{i_1 k 1})(b_{i_3 k_2} / b_{i_2 k_2})...(b_{i_n k_{n-1}} / b_{i_{n-1} k_{n-1}}) \tag{9}$$

where the right side of (9) is obtained by substitution from (8). Define $d_h = b_{i_h k_{h-1}} / b_{i_h k_h}$ for $h = 2, 3, ..., n$ and $d_1 = b_{i_1 k_n} / b_{i_1 k_1}$. From (9), we have

$$d_1 d_2 d_3 ... d_n = 1 \tag{10}$$

Define $$c_h = d_1 d_2 ... d_h \tag{11}$$

and $$D = \min\left(P_{i_1 k_1}/c_1, P_{i_2 k_1}/c_1, P_{i_2 k_2}/c_2, P_{i_3 k_2}/c_2, ..., P_{i_h k_h}/c_h, P_{i_{h+1} k_h}/c_h, ..., P_{i_n k_n}/c_n, P_{i_1 k_n}/c_n\right). \tag{12}$$

Suppose that $D = P_{i_h k_h}/c_h$ for some *h* (note: the case where $D = P_{i_{h+1} k_h}/c_h$ follows a similar probability-shifting procedure as below except that the + and – signs are reversed.). Then, shift probabilities as follows to obtain a new solution:

$$P_{i_h k_h} \leftarrow P_{i_h k_h} - c_h D = 0\,;\; P_{i_{h+1} k_h} \leftarrow P_{i_{h+1} k_h} + c_h D \geq 0\,;$$
$$P_{i_{h+1} k_{h+1}} \leftarrow P_{i_{h+1} k_{h+1}} - c_{h+1} D \geq 0\,;\; P_{i_{h+2} k_{h+1}} \leftarrow P_{i_{h+2} k_{h+1}} + c_{h+1} D \geq 0\,;$$
$$\vdots$$
$$P_{i_n k_n} \leftarrow P_{i_n k_n} - c_n D \geq 0\,;\; P_{i_1 k_n} \leftarrow P_{i_1 k_n} + c_n D \geq 0\,;$$
$$\vdots$$
$$P_{i_h k_{h-1}} \leftarrow P_{i_h k_{h-1}} - c_{h-1} D \geq 0$$

After applying the above procedure, the change in $T_{i_j}^*$ for $j = 2,...,n$ is

$$-b_{i_j k_j} c_j D + b_{i_j k_{j-1}} c_{j-1} D = 0\,; \tag{13}$$

and the change in $T_{i_1}^*$ is

$$-b_{i_1 k_1} c_1 D + b_{i_1 k_n} c_n D = -b_{i_1 k_1} d_1 D + b_{i_1 k_n} D = 0\,. \tag{14}$$

So, the new solution remains optimal. Furthermore, one edge in the bipartite graph, (i_h, k_h), has been removed.

If there is still a loop remaining, iterate the above procedure until no loop is left. When there is no loop left, there cannot be more than $(N + K - 1)$ edges. For this new solution, let K_s be the number of shared channels and K_u be the number of unshared channels. Each shared channel is associated with at least two edges; and each unshared channel is associated with one edge (note: an unshared channel cannot be associated with zero edge after the above loop-removal procedure because we started out assuming all the K channels are shared, and the loop-free procedure preserves $\sum_{i \in I_N} P_{ik}$ for each channel k, so that at least one user is still associated with channel k). So, $K_s \leq (N + K - 1 - K_u)/2$. This gives $K_s \leq N - 1$. □

B. Numbers of Multiply and Singly Assigned Users

Definition of Multiply Assigned Users:

1. A user *i* is said to be assigned to multiple channels (or multiply assigned) if there are at least two non-zero P_{ik}, $k \in K_S$, where K_S is the set of all channels in the system.
2. A user *i* is multiply assigned to *K* ($K \leq S$) channels in the system if there are at least two non-zero P_{ik}, $k \in K_K$, where $K_K \subseteq K_S$ is the subset containing the *K* channels.
3. *N* ($N \leq U$) users are multiply assigned to *K* ($K \leq S$) channels if each and every of the *N* users is multiply assigned to the *K* channels.

Theorem 4: Consider a system with U users and S channels. There is an optimal solution in which the number of users that are multiply assigned to any K of the S channels is no more than $K-1$.

Corollary 2: There is an optimal solution with no more than $\min(U, S-1)$ multiply assigned users in the overall system, and with at least $\max(0, U-S+1)$ users non-multiply assigned.

Comment**:** With respect to the AP allocation problem in Fig. 2, to the extent that there are many more STAs than APs, Corollary 2 basically says that most STAs will associate with only one AP.

Proof of Corollary 2: Obvious from Theorem 4.

Proof of Theorem 4: Similar to the proof of Theorem 3, we start out by assuming there are $N \geq K$ users that are multiply assigned to K channels in an optimal solution. We then find a loop-free optimal solution using the loop-removal procedure. So, the number of non-zero P_{ik} in the loop-free optimal solution is no more than $(N+K-1)$. Each user i must have at least one non-zero P_{ik} after the loop-removal procedure, since otherwise T_i cannot be preserved. Let N_m be the number of multiply assigned users and N_u be the number of non-multiply assigned users after the loop-removal procedure. Then,

$$2N_m + N_u \leq N + K - 1. \tag{15}$$

So, we have

$$N_m \leq K - 1 \tag{16}$$

after substituting

$$N = N_m + N_u.$$

□

IV. PROPORTIONAL-FAIRNESS ALGORTIHMS

This section presents several algorithms for the PF optimization problem. Subsections A and B consider the special cases where there are only 2 users and 2 channels, respectively. The optimal-solution characteristics derived in the preceding come in handy for the construction of fast algorithms in these cases. Subsection C presents a parallel algorithm for the general case.

A. 2-User-S-Channel Case

We present a fast $O(S \log S)$ algorithm for the 2-user-S-channel case. The idea is to sort the S channels according to $b_{1,k}/b_{2,k}$ from large to small. The $O(S \log S)$ computation time is due to the sort operation. Let us relabel the channel numbers according to the sort result so that $b_{1,k}/b_{2,k} \geq b_{1,(k+1)}/b_{2,(k+1)}$ for all k.

According to Corollary 1, there is an optimal solution with at most one channel that is shared by the two users. Together with the KKT conditions, this implies the following property:

Property 1: There is an optimal solution with throughputs T_1^* and T_2^*, and a channel S^*, such that either (i) all channels $k \le S^*$ are exclusively assigned to user 1; all channels $k > S^*$ are exclusively assigned to user 2; or (ii) channel S^* is shared; all channels $k < S^*$ are exclusively assigned to user 1; all channels $k > S^*$ are exclusively assigned to user 2.

For (i), $$T_1^* \big/ T_2^* \le b_{1,S^*} \big/ b_{2,S^*} \text{ and } T_1^* \big/ T_2^* \ge b_{1,(S^*+1)} \big/ b_{2,(S^*+1)} . \tag{17}$$

For (ii), $$T_1^* \big/ T_2^* = b_{1,S^*} \big/ b_{2,S^*} . \tag{18}$$

Let us define $T_1^{(k)} = \sum_{l=1}^{k} b_{1,l}$ and $T_2^{(k)} = \sum_{l=k+1}^{S} b_{2,l}$ (i.e., the solution given by exclusively assigning channels 1 to *k* to user 1, and channels *k*+1 to *S* to user 2). Note that

$$T_1^{(k)} \big/ T_2^{(k)} \le T_1^{(k+1)} \big/ T_2^{(k+1)} \quad \forall k . \tag{19}$$

Property 2: Suppose that S^l and S^u are some known lower and upper bounds for the optimal solution S^* (i.e., $S^l \le S^* \le S^u$). Consider a *tentative* solution S' within the bound, in which channels 1 to S' are exclusively assigned to user 1 and the other channels are exclusively assigned to user 2. (i) If $T_1^{(S')} \big/ T_2^{(S')} > b_{1,S'} \big/ b_{2,S'}$, then $S^* \le S'$. (ii) If $T_1^{(S')} \big/ T_2^{(S')} < b_{1,(S'+1)} \big/ b_{2,(S'+1)}$, then $S^* \ge S'+1$.

To see Property 2(i), consider a channel $k > S'$. Then,

$$\frac{b_{1,k}}{b_{2,k}} \le \frac{b_{1,S'}}{b_{2,S'}} < \frac{T_1^{(S')}}{T_2^{(S')}} \le \frac{T_1^{(k)}}{T_2^{(k)}} . \tag{20}$$

The inequality $\frac{b_{1,k}}{b_{2,k}} < \frac{T_1^{(k)}}{T_2^{(k)}}$ means k cannot be S^* under solution (i) in Property 1. That $\frac{b_{1,k}}{b_{2,k}} < \frac{T_1^{(S')}}{T_2^{(S')}} \le \frac{T_1^{(k)}}{T_2^{(k)}}$ for $k > S'$ means k cannot be S^* under solution (ii) in Property 1 either, since $T_1^* \big/ T_2^* = b_{1,S^*} \big/ b_{2,S^*}$ cannot be achieved by shifting probability from $P_{1,k}$ to $P_{2,k}$.

To see Property 2(ii), consider a channel $k \le S'$. Then,

$$\frac{b_{1,k}}{b_{2,k}} \ge \frac{b_{1,(S'+1)}}{b_{2,(S'+1)}} > \frac{T_1^{(S')}}{T_2^{(S')}} \ge \frac{T_1^{(k)}}{T_2^{(k)}} . \tag{21}$$

The inequality $\frac{b_{1,k}}{b_{2,k}} > \frac{T_1^{(k)}}{T_2^{(k)}}$ means k cannot be S^* under solution (ii) in Property 1 by shifting probability from $P_{1,k}$ to $P_{2,k}$. Also,

$$\frac{b_{1,(k+1)}}{b_{2,(k+1)}} \geq \frac{b_{1,(S'+1)}}{b_{2,(S'+1)}} > \frac{T_1^{(S')}}{T_2^{(S')}} \geq \frac{T_1^{(k)}}{T_2^{(k)}}. \quad (22)$$

So, k cannot be S^* under solution (i) in Property 1 either.

The following is a binary search algorithm to identify S^* based on Properties 1 and 2.

2-User-S-Channel Algorithm

Initial solution: $S' \leftarrow \lfloor S/2 \rfloor$; $S^l \leftarrow 1$; $S^u \leftarrow S$.

Step 1: $P_{1,k} \leftarrow 1$ for $k = 1,...,S'$; $P_{2,k} \leftarrow 1$ for $k = S'+1,...,S$; $P_{i,k} \leftarrow 0$ otherwise.
Compute $T_1^{(S')}$ and $T_2^{(S')}$.

Step 2: if $T_1^{(S')} / T_2^{(S')} > b_{1,S'} / b_{2.S'}$ (*See Property 2(i)*)
then { $S^u \leftarrow S'$; $S' \leftarrow \lfloor (S^l + S^u)/2 \rfloor$;
if $S^u = S^l$ then goto step 5; else goto Step 1. }

Step 3: if $T_1^{(S')} / T_2^{(S')} < b_{1,(S'+1)} / b_{2,(S'+1)'}$ (*see Property 2(ii)*)
then { $S^l \leftarrow S'+1$; $S' \leftarrow \lfloor (S^l + S^u)/2 \rfloor$;
if $S^u = S^l$ then goto Step 5; else goto Step 1. }

Step 4: $S^* \leftarrow S'$; (*Condition in Property 1(i) satisfied. All channels exclusively assigned.*)
stop.

Step 5: $S^* \leftarrow S'$; *(Channel S^* is shared.)*
$P_{1,k} \leftarrow 1$ for $k = 1,...,S^* - 1$ and $P_{2,k} \leftarrow 1$ for $k = S^* + 1,...,S$;

$$P_{1S^*} \leftarrow \frac{1 + (b_{2,S^*+1} + ... + b_{2,S}) / b_{2,S^*} - (b_{1,1} + ... + b_{1,S^*-1}) / b_{1,S^*}}{2};$$

$P_{2S^*} \leftarrow 1 - P_{1S^*}$;
stop.

B. U-User-2-Channel Case

We present a fast $O(U \log U)$ algorithm for the U-user-2-channel case. The idea is similar to the 2-user-S-channel case. We sort the U users according to $b_{i,1}/b_{i,2}$ from large to small, and relabel the user index so that $b_{i,1}/b_{i,2} \geq b_{(i+1),1}/b_{(i+1),2}$ for all i. According to Corollary 2, there is an optimal solution with at most one user that uses both channels. The rest use just one of the channels.

Property 3: To achieve proportional fairness, equal airtime should be assigned to the users that are non-multiply assigned to the same channel.

Corollary 2, the KKT conditions, and Property 3 imply the following property:

Property 4: There is an optimal solution with throughputs T_i^*, $i=1,...,U$, and a user U^*, such that either (i) all users $i \le U^*$ use channel 1 only; all users $i > U^*$ use channel 2 only; or (ii) user U^* use both channels; all users $i < U^*$ use channel 1 only; all users $i > U^*$ use channel 2 only.

For (i),
$$\frac{b_{(U^*+1),1}}{b_{(U^*+1),2}} \le \frac{U^*}{U-U^*} \le \frac{b_{U^*,1}}{b_{U^*,2}}. \tag{23}$$

For (ii),
$$\frac{U^*-1}{U-U^*+1} < \frac{b_{U^*,1}}{b_{U^*,2}} < \frac{U^*}{U-U^*}. \tag{24}$$

(i) in the above is derived from the fact that $T_{U^*}^* \big/ T_{U^*+1}^* \le b_{U^*,1} \big/ b_{(U^*+1),1}$, $T_{U^*}^* \big/ T_{U^*+1}^* \ge b_{U^*,2} \big/ b_{(U^*+1),2}$, and Property 3 (i.e., $T_{U^*}^* = b_{U^*,1} \big/ U^*$ and $T_{U^*+1}^* = b_{U^*+1,2} \big/ (U-U^*)$. To see (ii), user U^* uses channels 1 and 2 with probabilities $P_{U^*,1}$ and $P_{U^*,2}$, respectively. By computing T_i^* for all *i*, and setting

$$T_{U^*}^* \big/ T_i^* = b_{U^*,1} \big/ b_{i,1} \text{ for } i=1,...,U^*-1 \tag{25}$$

and
$$T_{U^*}^* \big/ T_i^* = b_{U^*,2} \big/ b_{i,2} \text{ for } i=U^*+1,...,U, \tag{26}$$

we can find an expression for $P_{U^*,1}$ and an expression for $P_{U^*,2}$. The requirements of $P_{U^*,1} > 0$ and $P_{U^*,2} > 0$ lead to (ii).

Property 5: Suppose that U^l and U^u are some known lower and upper bounds for the optimal solution U^* (i.e., $U^l \le U^* \le U^u$). Consider a *tentative* solution U' within the bound, in which users 1 to U' use only channel 1 and the other users use only channel 2. (i) If $\frac{b_{U',1}}{b_{U',2}} < \frac{U'}{U-U'}$, then $U^* \le U'$.

(ii) If $\frac{b_{(U'+1),1}}{b_{(U'+1),2}} > \frac{U'}{U-U'}$, then $U^* \ge U'+1$.

To see Property 5(i), consider a user $i \ge U'+1$. Then,

$$\frac{i}{U-i} > \frac{i-1}{U-i+1} \ge \frac{U'}{U-U'} > \frac{b_{U',1}}{b_{U',2}} \ge \frac{b_{i,1}}{b_{i,2}}. \tag{27}$$

So, *i* cannot be U^* under solutions (i) or (ii) in Property 4. To see Property 5(ii), consider a user $i \le U'$. Then,

$$\frac{i}{U-i} \le \frac{U'}{U-U'} < \frac{b_{(U'+1),1}}{b_{(U'+1),2}} \le \frac{b_{(i+1),1}}{b_{(i+1),2}} \le \frac{b_{i,1}}{b_{i,2}}. \tag{28}$$

So, *i* cannot be U^* under solutions (i) or (ii) in Property 4 either.

U-User-2-Channel Algorithm

Initial solution: $U' \leftarrow \lfloor U/2 \rfloor$; $U^l \leftarrow 1$; $U^u \leftarrow U$.

Step 1: if $\frac{b_{U',1}}{b_{U',2}} < \frac{U'}{U-U'}$ (*see Property 5(i)*)

then { $U^u \leftarrow U'$; $U' \leftarrow \lfloor (U^l + U^u)/2 \rfloor$;

if $U^l = U^u$ then goto Step 4; else goto Step 1}

Step 2: if $\frac{b_{(U'+1),1}}{b_{(U'+1),2}} > \frac{U'}{U-U'}$ (*see Property 5(ii)*)

then { $U^l \leftarrow U'+1$; $U' \leftarrow \lfloor (U^l + U^u)/2 \rfloor$;

if $U^l = U^u$ then goto Step 4; else goto Step 1}

Step 3: $U^* \leftarrow U'$; (*Condition in Property 4(i) satisfied. All users non-multiply assigned.*)

$$P_{i,1} \leftarrow \begin{cases} \frac{1}{U^*} & \text{for } i = 1,\cdots,U^* \\ 0 & \text{otherwise} \end{cases}; \quad P_{i,2} \leftarrow \begin{cases} \frac{1}{U-U^*} & \text{for } i = U^*+1,\cdots,U \\ 0 & \text{otherwise} \end{cases}.$$

stop.

Step 4: $U^* \leftarrow U'$; (*User U^* uses both channels*)

$$P_{i,1} \leftarrow \begin{cases} \frac{U-U^*+1}{U} - \frac{U^*-1}{U} \cdot \frac{b_{U^*,2}}{b_{U^*,1}} & \text{for } i = U^* \\ \frac{1-P_{U^*,1}}{U^*-1} & \text{for } i = 1,\cdots,U^*-1 \\ 0 & \text{otherwise} \end{cases}; \quad P_{i,2} \leftarrow \begin{cases} \frac{U^*}{U} - \frac{U-U^*}{U} \cdot \frac{b_{U^*,1}}{b_{U^*,2}} & \text{for } i = U^* \\ \frac{1-P_{U^*,2}}{U-U^*} & \text{for } i = U^*+1,\cdots,U \\ 0 & \text{otherwise} \end{cases}$$

stop.

C. U-User-S-Channel Case

We now present a parallel algorithm for the general *U*-user-*S*-channel case. In the algorithm, $P_{i,k}$ is adjusted step by step. In each step, for each channel *k* we try to equate $b_{i,k}/T_{i,k}$ for all users with $P_{i,k} > 0$, so that the KKT condition is satisfied. The computation-intensive steps of the algorithm below (steps marked with *) can be carried out on all channels in parallel for fast execution speed. To start with, we set the initial $P_{i,k}$ to be $1/U$ for all *i* and *k*. The algorithm, however, will work for other initial $P_{i,k}$. To avoid oscillations, we use a factor ς to limit the maximum step size by which $P_{i,k}$ can be adjusted in each iteration.

U-User-S-Channel Algorithm

Without loss of generality, we focus on an arbitrary channel *k* in the following description.

Initial solution: $P_{i,k} \leftarrow 1/U \ \forall i$; $T_i \leftarrow \sum_{k=1}^{S} b_{i,k} \Big/ U \ \forall i$.

Step 1*: $I_k \leftarrow \{i \mid P_{i,k} > 0\}$; $R_k = \sum_{i \in I_k} \frac{b_{i,k}}{T_i} \Big/ |I_k|$.

(Note that $dy/dP_{i,k} = b_{i,k}/T_i$ and R_k as computed above is the average $dy/dP_{i,k}$ among all users with $P_{i,k} > 0$. The parameter R_k serves as a "reference $dy/dP_{i,k}$" in our algorithm such that users with $dy/dP_{i,k} \ge R_k$ will have their $P_{i,k}$ increased, while users with $dy/dP_{i,k} < R_k$ and $P_{i,k} > 0$ will have their $P_{i,k}$ decreased.)

Step 2: $I_k^+ \leftarrow \left\{ i \mid i \in I_k \text{ and } \frac{b_{i,k}}{T_i} \ge R_k \right\}$; $I_k^- \leftarrow \left\{ i \mid i \in I_k \text{ and } \frac{b_{i,k}}{T_i} < R_k \right\}$;

$\bar{I}_k^+ \leftarrow \left\{ i \mid P_{i,k} = 0 \text{ and } \frac{b_{i,k}}{T_i} \ge R_k \right\}$; $\bar{I}_k^- = \left\{ i \mid P_{i,k} = 0 \text{ and } \frac{b_{i,k}}{T_i} < R_k \right\}$

$R_k^{new} \leftarrow \sum_{i \in I_k \cup \bar{I}_k^+} \frac{b_{i,k}}{T_i} \Big/ \left| I_k \cup \bar{I}_k^+ \right|$;

if $R_k = R_k^{new}$, then goto Step 3

else $R_k \leftarrow R_k^{new}$ and goto Step 2.

(*The purpose of Step 2 is include users with $P_{i,k} = 0$ but $dy/dP_{i,k} \ge R_k$ (i.e., $\bar{I}_k^+$) in the set of user whose $P_{i,k}$ will be increased. Step 1 included only users with $P_{i,k} > 0$ as a first attempt.* Note that R_k is adjusted to be the average $dy/dP_{i,k}$ of the users whose $P_{i,k}$ will be increased.)

Step 3*: $\delta_{i,k} \leftarrow \frac{b_{i,k}}{T_i} - R_k \ \forall i \in I_k \cup \bar{I}_k^+$; $\delta_{i,k} \leftarrow 0 \ \forall i \in \bar{I}_k^-$;

$$\alpha \leftarrow \min_{i \in I_k^+ \cup \bar{I}_k^+} \frac{1 - P_{i,k}}{\delta_{i,k}}; \ \beta \leftarrow \min_{i \in I_k^-} \frac{-P_{i,k}}{\delta_{i,k}}; \ c \leftarrow \min(\varsigma, \alpha, \beta); \quad (29)$$

$$P_{i,k} \leftarrow P_{i,k} + c\delta_{i,k} \ \forall i \quad (30)$$

(*The amount by which $P_{i,k}$ will be increased (decreased) is proportional to $\delta_{i,k}$, with constant of proporationality c. It is easy to show that $\sum_i c\delta_{i,k} = 0$ and $\sum_i \left(P_{i,k} + c\delta_{i,k}\right) = 1$. In other words, the airtime reallocation in (30) does not change the total airtime usage.*

Note that α and β in (29) is to ensure the new probability assignment stays between 0 and 1 (i.e., $0 \le P_{i,k} + c\delta_{i,k} \le 1 \forall i$). The parameter ς imposes a limit on the adjustment of $P_{i,k}$ in each iteration to avoid oscillations.)

Step 4: $T_i \leftarrow \sum_{k=1}^{S} P_{i,k} b_{i,k} \ \forall i$;

if the KKT condition is satisfied, (i.e., $\frac{b_{i,k}}{T_i} = \frac{b_{j,k}}{T_j}$ if $P_{i,k} > 0$ and $P_{j,k} > 0$ and $\frac{b_{i,k}}{T_i} \ge \frac{b_{j,k}}{T_j}$ if $P_{i,k} > 0$ and $P_{j,k} = 0 \quad \forall k$), then stop;

else goto Step 1.

The above *U*-user-*S*-channel algorithm is used to generate numerical results for the study in the next section. A non-parallel version of the program has been written using MATLAB. Alternatively, the built-in functions in MATLAB optimization toolbox based on generic algorithms[1] could be used. However, we find that using the generic algorithms takes exceedingly long computational time even for PF-optimization problems of moderate size, making generating a large number of data points for the numerical study in the next section virtually impossible. In contrast, the computational time is quite manageable with the above *U*-user-*S*-channel algorithm, even for a non-parallel version. It typically takes around 0.6 seconds[2] to converge when there are, for instance, 16 APs and 64 mobile stations in the WiFi network shown in Fig. 2. With this kind of time scale, the algorithm is also suitable for actual field deployment beyond mere numerical studies, since AP allocation and re-allocation are usually not invoked in a frequent manner in typical WLAN-usage scenarios where the users are not highly mobile.

V. NUMERICAL RESULTS: PF CAPACITY ALLOCATION IN WIFI NETWORKS

We now move on to the capacity allocation problem in WiFi networks with multiple adjacent WLANs (see Fig. 2). In this study, we assume that there are 16 APs being placed in a square grid. The adjacent APs are separated by 20 meters. A wrap-around method is applied to create a torus topology to eliminate the edge effect: i.e., the rightmost column (top row) is adjacent to the leftmost column (bottom row). A mobile station can transmit at different data rates depending on the SNR with respect to an AP. The possible data transmission rates and the corresponding required SNRs are listed in Table 1.

[1] MATLAB solves the constrained nonlinear optimization problem using a subspace trust region method for large-scale problems and a sequential quadratic programming method for medium scale problems.
[2] We performed our simulations with MATLAB 7.0 on a Pentium 2GHz machine.

We further assume a two-ray ground model with path loss exponent of 3 and log-normal shadowing with standard deviation of 6 dB. The average SNR (averaged over shadowing) at the cell boundary is 10 dB. That is, there is a 4 dB shadowing margin for achieving a minimum data rate of 1Mbps.

For comparison purposes, besides PF, three conventional AP association schemes, namely, maximum throughput (MT), signal-strength based association with intra-cell throughput fairness (SS-TF), and signal-strength based association with intra-cell airtime fairness (SS-AF), are also simulated.

MT aims to maximize the total throughput of the WLAN. Each AP selects among all the STAs those that enjoy the highest data transmission rate to serve. If more than one STA has the same highest rate, equal airtime is assigned to these STAs. SS-TF is adopted in the current 802.11 networks. The STAs associate themselves with the APs with the strongest signal. Meanwhile, the same throughput is guaranteed for the STAs associated with the same AP. SS-AF is similar to SS-TF except that the STAs associated with the same AP are allocated equal airtime. As proved in [3], intra-cell equal airtime allocation leads to PF optimality within a single AP coverage.

In the first set of experiments, we assume that the STAs are uniformly distributed in the whole area. Fig. 4 plots total throughput versus number of STAs in the overall network. An interesting observation is that when the number of STAs is small relative to the number of APs, the throughput of PF converges to that of MT. This is because most of the APs are exclusively allocated to just one STA in this case (see Theorem 3 and Corollary 1). To maximize the PF utility, the STA chosen by an AP is the one with the highest throughput, which coincides with MT.

In contrast, when the total number of STAs is much larger than the number of APs, the throughput of PF converges to that of SS-AF. This is also due to the characteristic of PF optimal solutions (see Theorem 4 and Corollary 2). When there are many more STAs than APs, most STAs are associated with only one AP, which is usually the one with the strongest signal strength. Meanwhile, PF optimality leads to equal airtime allocation within each cell, which coincides with SS-AF.

Fig. 4 also indicates that SS-AF and PF outperform SS-TF. Most current WiFi products adopt SS-TF, in which (i) each STA associates with the AP with the highest signal strength; and (ii) the default 802.11 MAC scheduling algorithm is used. An STA at cell boundary has weak SNR and transmits at low data rates. With SS-TF, the throughputs of all STAs will be dragged down by these “weak STAs” [1]. With SS-AF, (ii) is modified to ensure equal airtime for all STAs of an AP [3]. The equal airtime allocation establishes a “firewall” between the strong and weak STAs so that the weak STAs do not eat into the airtime of the strong STAs. We also note that whereas SS-AF is better than SS-TF only when number of STAs is large, PF is better than SS-TF for both small and large numbers of STAs.

In Table 2, we compare the fairness performance of the PF scheme with other schemes using the Jain's fairness index [6]:

$$\left|\sum_{i=1}^{U} T_i\right|^2 \Big/ U\sum_{i=1}^{U}(T_i)^2 , \qquad (31)$$

We see that the fairness of MT is significantly worse than the other schemes. Comparatively, PF, SS-TF and SS-AF guarantee much fairer service. In particular, PF achieves consistently better fairness than MT, SS-TF, and SS-AF do.

In Fig. 5, we investigate the outage probability. A user is said to be suffering an outage if its throughput is lower than a minimum data-rate requirement, which is assumed to be 1Mbps in the figure. As the figure shows, PF achieves the lowest outage probability among the four schemes.

In Fig. 4, Fig. 5, and Table 2, we have demonstrated that PF strikes a good balance between system throughput and fairness. In the following figures, we show that in a WLAN with hot spots, PF can effectively balance traffic loads among the cells. In this set of experiments, the total number of STAs is 64. Out of the 16 APs, one AP is a hot spot. We define the load percentage of the hot spot to be the percentage of users that are located in the hot spot. The users that are not located in the hot spot are randomly distributed in the other cells. We vary the load percentage of the hot spot from 6.25% (i.e., 1/16, which corresponds to uniform STA distribution) to 100%.

A high STA density in the hot spot inevitably results in high outage probability. Fig. 6 shows that PF can mitigate this destructive effect. In particular, unlike the other schemes, its outage probability increases by 3.50% only when the traffic distribution varies from uniform to extremely non-uniform. Fig. 7 illustrates the throughput degradation in the presence of non-uniform traffic distribution. From Fig. 6 and Fig. 7, we can see that PF achieves both higher throughput and lower outage probability compared with SS-TF. Moreover, PF outperforms SS-AF in terms of throughput when the load percentage of the hot spots exceeds 80%.

VI. CONCLUSIONS

This paper has (i) provided physical/economic interpretations for the use the proportional-fairness (PF) utility function for capacity allocation in multi-channel multi-rate wireless networks; (ii) derived characteristics of PF optimal solutions and presented several PF algorithms; and (iii) investigated the use of PF and other utility functions for capacity allocation and AP assignment in large-scale WiFi networks.

With regard to (i), we have shown by general mathematical arguments that PF optimization leads to equal airtime allocation to individual users for the single-channel case; and equal *equivalent airtime* allocation to individual users for the multi-channel case, where the equivalent airtime enjoyed by a user is defined to be a weighted sum of the airtimes enjoyed by the user on all channels, with the weight of a channel being the *price* or *value* of that channel. We have also established the Pareto efficiency of PF-optimal solutions. In addition, we have derived several characteristics of PF-optimal solutions that are useful for the construction of PF-optimization algorithms.

With regard to (ii), we show that a PF solution typically consists of many zero airtime assignments when the difference between the number of users U and the number of channels S, $|U-S|$, is large. We have applied this property to construct fast algorithms for the 2-user-S-channel and U-user-2-channel cases. In addition, we have presented a fast algorithm amenable to parallel implementation for the general U-user-S-channel case.

With regard to (iii), we have found that using the PF utility function achieves a good balance between system throughput and fairness compared with using the other utility functions. In particular, PF simultaneously achieves higher system throughput, better fairness, and lower outage probability with respect to the default 802.11 AP association and MAC scheduling scheme in today's commercial products. This is the case for uniform as well as non-uniform, and dense as well as sparse, user distributions in the wireless network.

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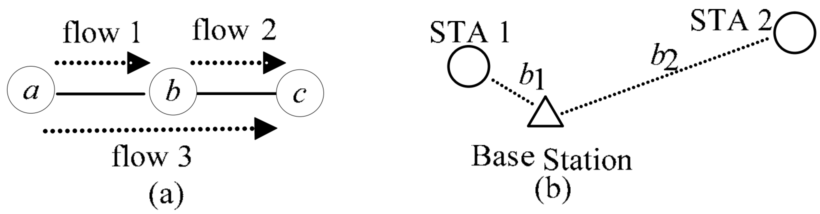

Fig 1: Illustrating examples of trade-off between throughput and fairness in (b) a wired network, and (b) a wireless network.

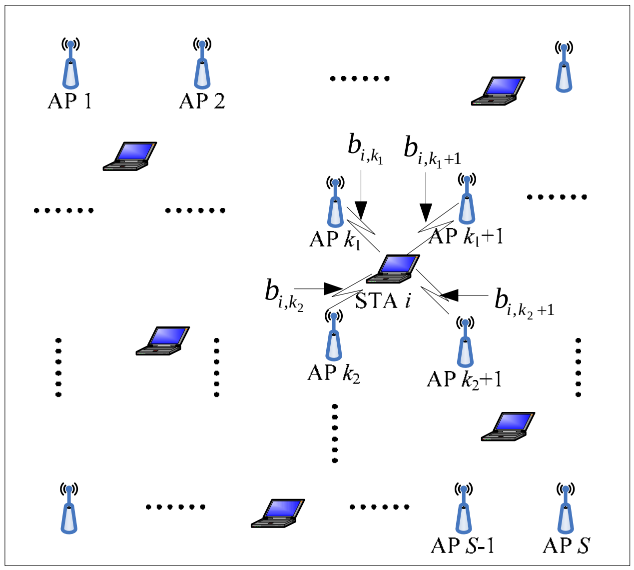

Fig. 2: Assigning APs to wireless stations in a wireless LAN

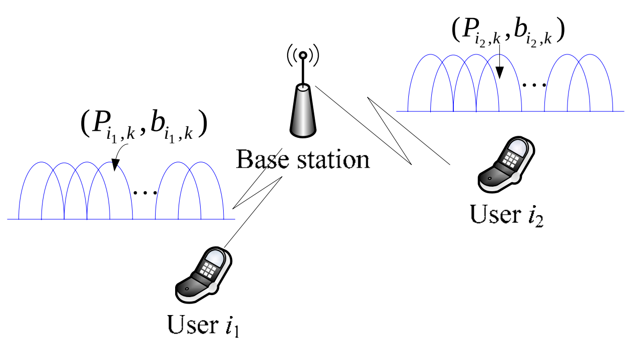

Fig. 3: Assigning subcarriers to users in an OFDM system

Table 1: Minimum SNR required for different data transmission rates

Data Rate (Mbps)	Minimum required SNR (dB)
0	$-\infty$
1	6
6	10
9	11
12	12

18	13
24	16
36	19
48	26
54	29

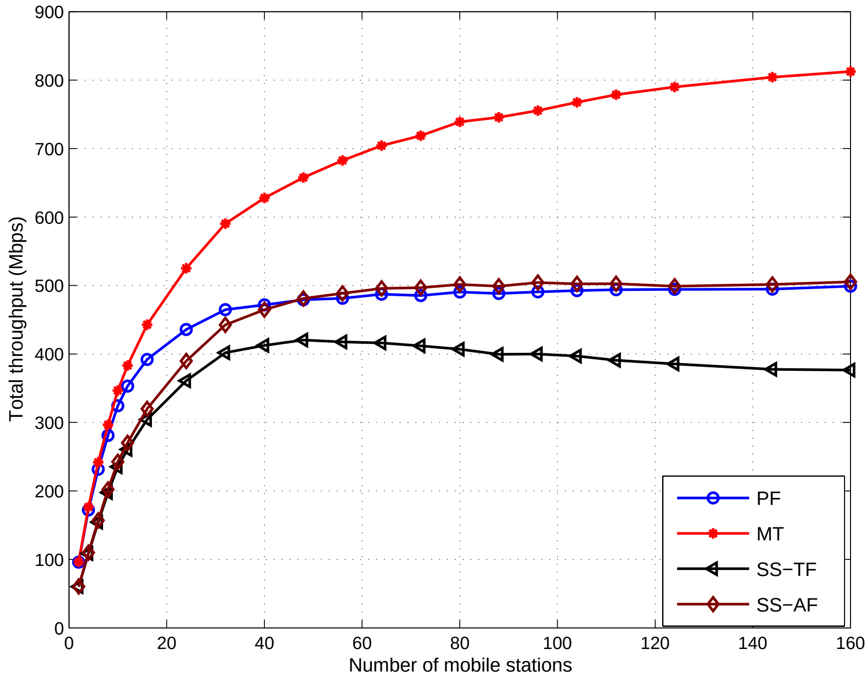

Fig. 4: Total Throughput for uniform STA distribution.

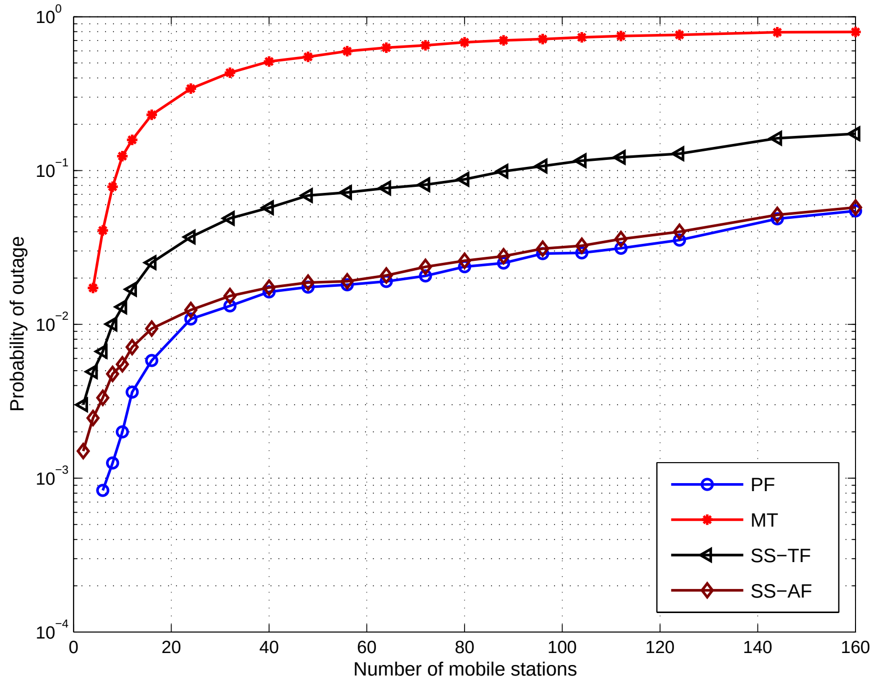

Fig. 5: Probability of outage for uniform STA distribution.

Table 2: Jain's Fairness index

	U=32	*U*=48	*U*=64
PF	0.759	0.779	0.797
MT	0.432	0.291	0.277
SS-TF	0.612	0.604	0.635
SS-AF	0.649	0.639	0.661

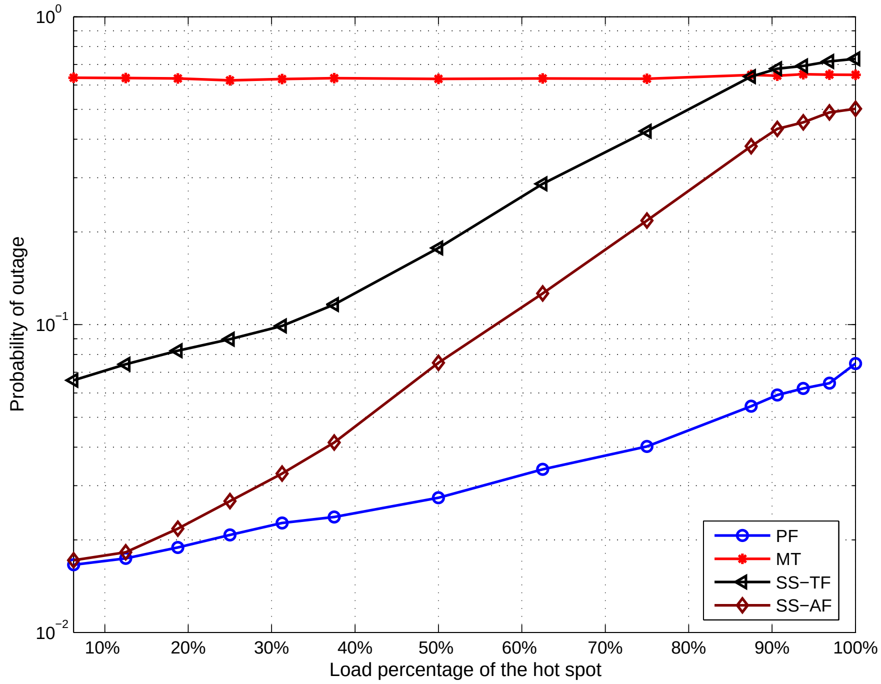

Fig. 6: Probability of outage for non-uniform STA distribution. The minimum data-rate requirement is 1Mbps

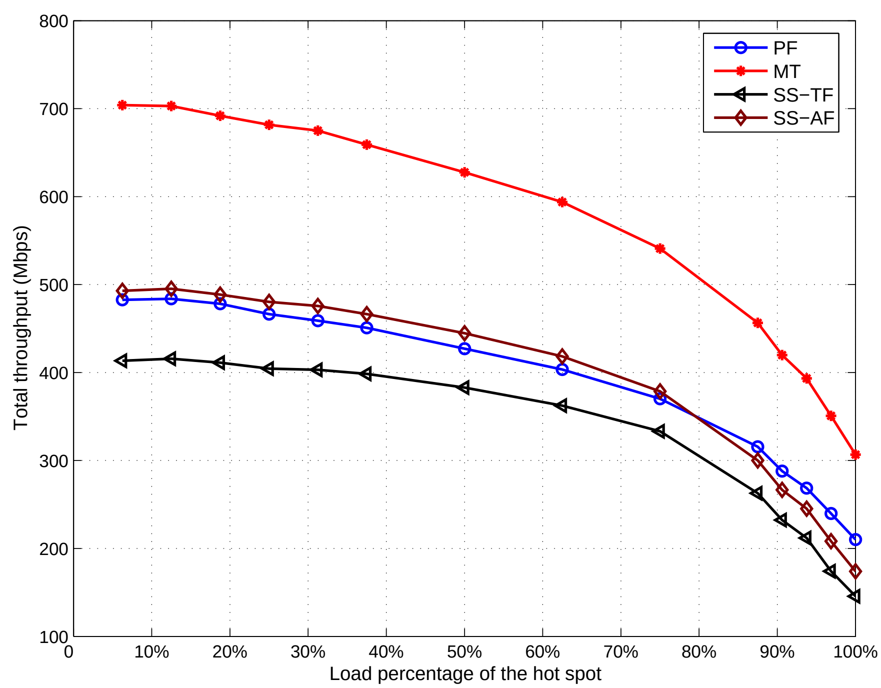

Fig. 7: Probability of outage for non-uniform STA distribution