# A Conjugate Augmented Approach to Direction-of-Arrival Estimation

Zhilong Shan and Tak-Shing P. Yum, *Senior Member, IEEE*

***Abstract*—In this paper, we propose a new Direction-of-Arrival (DOA) estimator called Conjugate Augmented MUSIC (CAM). The basic idea of CAM is to use the second-order statistics of the received signals to get the conjugate steering matrix. This, together with the steering matrix, is used to find the fourth-order cumulants. From that the source directions are obtained using the MUSIC-like algorithm. CAM can resolve two times the number of directions when compared to MUSIC-like estimator. Moreover, simulation results show that the estimation capacity, angle resolution, immunity to noise, and the number of required snapshots are all better than MUSIC-like algorithm.**

***Index Terms*—Array signal processing, DOA, fourth-order cumulants, MUSIC-like estimator, non-Gaussian sources.**

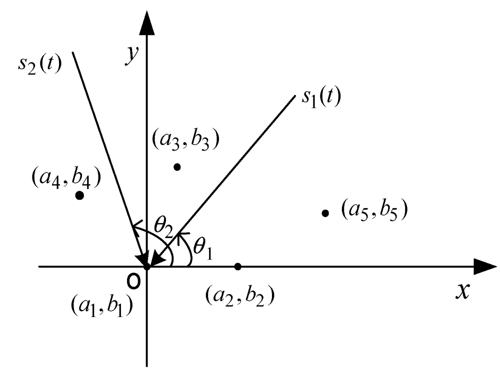


Fig. 1. Arbitrary array with five sensors.

## I. Introduction

IN recent years, the estimation of direction-of-arrival (DOA) is a hot topic in array signal processing because of its important applications in radar and wireless location. Among the methods proposed, the signal subspace algorithms have attracted a lot of interest due to the introduction of MUSIC algorithm [1], [2]. However, MUSIC and modified versions of MUSIC require the noise characteristics of the sensors be known and the total number of signals impinging on the array be less than the number of sensors [3].

If a non-Gaussian signal is received along with additive Gaussian noise, the method proposed by Porat and Friedlander [4] [which uses fourth-order (FO) cumulants] can be used to eliminate the effect of Gaussian noise. For convenience, the method in [4] is denoted as MUSIC-like estimator in this paper. With FO cumulants, a physical array can be extended to a larger size virtual array [5] and allows a greater number of signals to be estimated. Thus, for the MUSIC-like estimator, an array of $M$ identical physical sensors can be extended to a maximum of $M^2 - M + 1$ virtual sensors [6]. For a uniform linear array (ULA), the number of virtual sensors is showed in [6] to be $2M - 1$.

In this paper, we present a new estimator called Conjugate Augmented MUSIC (CAM). The basic idea is to use temporal information in addition to spatial information when estimating directions. After computing the second-order statistics of the received signals to get the conjugate steering matrix, a new conjugate augmented steering matrix is constructed. We then extract the FO cumulants of this new matrix for estimating the source directions. The estimation algorithm is presented in Section III. Then, in Section IV, the estimation capacity of CAM is derived. We show that CAM can estimate two times the number of directions when compared with the MUSIC-like estimator. We then apply CAM to ULA and show in Section V that the complexity of CAM/ULA is significantly smaller than CAM. In Section VI, CAM is compared to MUSIC-like in run time, estimation capacity, angle resolution, immunity to noise, and the number of required snapshots by computer simulation. Section VII concludes the paper.

Manuscript received December 18, 2003; revised January 11, 2005. This work was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China, under Project CUHK4220/03E. The associate editor coordinating the review of this paper and approving it for publication was Dr. Jan C. de Munck.

Z. Shan is with the School of Computers, South China Normal University, Guangzhou 510631, Guangdong, China (e-mail: sunnySZL@163.com).

T.-S. P. Yum is with the Department of Information Engineering, the Chinese University of Hong Kong, Shatin, Hong Kong (e-mail: tsyum@ie.cuhk.edu.hk).

Digital Object Identifier 10.1109/TSP.2005.857012

## II. Background

Consider $D$ narrowband plane wave signals impinging on an array of $M$ identical omnidirectional sensors. Let the signal from the $i$th source be denoted as $s_i(t) = A_i e^{j(\omega_c + \omega_i)t}$, where $\omega_c$ is the carrier frequency, $\omega_i$ is a small frequency offset for the $i$th signal with $\omega_i \ll \omega_c$, and $A_i$ is the amplitude of the $i$th signal [7]. After demodulation to IF, the signal due to the $i$th source becomes $s_i(t) = A_i e^{j\omega_i t}$. We assume that source signals are mutually independent, that the noises are also statistically independent to the signals, and that the array is in the same plane as the signals. We further assume that all sensors have the same linear time-invariant response and, hence, the same radiation patterns. Fig. 1 shows an example of sensor locations. Let $(a_m, b_m)$ be the coordinates of sensor $m$, and let the first sensor be located at the origin, i.e., $(a_1, b_1) = (0, 0)$. We use $[\cdot]^*$, $[\cdot]^{\mathrm{T}}$, and $[\cdot]^{\mathrm{H}}$ to denote the conjugate, the transpose, and the conjugate transpose, respectively. Let

- $s_i(t)$ be the zero-mean non-Gaussian signal from source $i$ of the form $s_i(t) = A_i e^{j\omega_i t}$ and $\mathbf{S}(t) = [s_1(t), \ldots, s_D(t)]^T$;

- $n_m(t)$ be the zero-mean Gaussian noise with variance $\sigma^2$ from sensor $m$ and $\mathbf{n}(t) = [n_1(t), \dots, n_M(t)]^T$;
- $\lambda$ be the wavelength of the carrier.

Then, the signal received at the $m$th ($m = 1, 2, \dots, M$) sensor can be expressed as

$$x_m(t) = \sum_{i=1}^{D} s_i(t) \exp\left\{-j\frac{2\pi}{\lambda}\left(a_m \cos(\theta_i) + b_m \sin(\theta_i)\right)\right\} + n_m(t). \quad (1)$$

In matrix form, it becomes

$$\mathbf{X}(t) = [x_1(t), \dots, x_M(t)]^T = \mathbf{AS}(t) + \mathbf{n}(t). \quad (2)$$

In (2), $\mathbf{A}$ is the steering matrix, which is defined as

$$\mathbf{A} = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_D)] \quad (3)$$

where $\mathbf{a}(\theta_i)$ is the steering vector associated with the $i$th source

$$\mathbf{a}(\theta_i) = \begin{bmatrix} 1 \\ \exp\left(-j\frac{2\pi}{\lambda}\left(a_2\cos(\theta_i) + b_2 \sin(\theta_i)\right)\right) \\ \vdots \\ \exp\left(-j\frac{2\pi}{\lambda}\left(a_M\cos(\theta_i) + b_M \sin(\theta_i)\right)\right) \end{bmatrix}. \quad (4)$$

For ULA, $\mathbf{A}$ is a Vandermonde matrix. Therefore, the columns of $\mathbf{A}$ are linear independent. For arbitrary array, $\mathbf{A}$ is known to be independent for MUSIC and MUSIC-like estimators. For symmetrically distributed signals, the FO cumulants of the sensor outputs for $k_1, k_2, k_3, k_4 \in \{1, 2, \dots M\}$ was derived for the MUSIC-like estimator in [4] as

$$\begin{aligned} &\text{cum}\left\{x_{k_1}, x_{k_2}^*, x_{k_3}^*, x_{k_4}\right\} \\ &\quad = E\left\{x_{k_1}, x_{k_2}^*, x_{k_3}^*, x_{k_4}\right\} - E\left\{x_{k_1}, x_{k_2}^*\right\} \\ &\quad \cdot E\left\{x_{k_3}^*, x_{k_4}\right\} - E\left\{x_{k_1}, x_{k_3}^*\right\} \cdot E\left\{x_{k_2}^*, x_{k_4}\right\}. \end{aligned} \quad (5)$$

These cumulants can be expressed in a matrix as

$$\begin{aligned} \mathbf{C_X} = &E\left\{(\mathbf{X}\otimes\mathbf{X}^*)(\mathbf{X}\otimes\mathbf{X}^*)^H\right\} - E\{\mathbf{X}\otimes\mathbf{X}^*\} \\ &\cdot E\left\{(\mathbf{X}\otimes\mathbf{X}^*)^H\right\} - E\{\mathbf{XX}^H\} \otimes E\left\{(\mathbf{XX}^H)^*\right\} \end{aligned} \quad (6)$$

where $\otimes$ denotes the Kronecker product, and cum $\{x_{k_1}, x_{k_2}^*, x_{k_3}^*, x_{k_4}\}$ appears as the $[(k_1-1)M + k_2]$th row and $[(k_3-1)M + k_4]$th column of $\mathbf{C_X}$. Similarly, the FO cumulants matrix of $\mathbf{S}$ can be written as

$$\begin{aligned} \mathbf{C_S} = &E\left\{(\mathbf{S}\otimes\mathbf{S}^*)(\mathbf{S}\otimes\mathbf{S}^*)^H\right\} - E\{\mathbf{S}\otimes\mathbf{S}^*\} \\ &\cdot E\left\{(\mathbf{S}\otimes\mathbf{S}^*)^H\right\} - E\{\mathbf{SS}^H\} \otimes E\left\{(\mathbf{SS}^H)^*\right\}. \end{aligned} \quad (7)$$

Since the FO cumulants of the Gaussian noise are identically zero and $\mathbf{C_S}$ is a diagonal matrix (the sources are independent), we get

$$\begin{aligned} \mathbf{C_X} &= (\mathbf{A}\otimes\mathbf{A}^*)C_\mathbf{S}(\mathbf{A}\otimes\mathbf{A}^*)^H \\ &= \sum_{i=1}^{D} \left(\mathbf{a}(\theta_i)\otimes\mathbf{a}^*(\theta_i)\right)\kappa_i\left(\mathbf{a}(\theta_i)\otimes\mathbf{a}^*(\theta_i)\right)^H \\ &= \mathbf{B\Sigma_S B}^H \end{aligned} \quad (8)$$

where $\kappa_i = \text{cum}\{s_i, s_i^*, s_i^*, s_i\}$ are the FO cumulants of $s_i$, $\mathbf{\Sigma_S}$ is a $D \times D$ diagonal matrix with the form

$$\mathbf{\Sigma_S} = \text{diag}\{\kappa_1, \dots, \kappa_D\} \quad (9)$$

and

$$\mathbf{B} = [\mathbf{b}(\theta_1), \dots, \mathbf{b}(\theta_D)] \quad (10)$$

where $\mathbf{B}$ is a $M^2 \times D$ matrix, and

$$\mathbf{b}(\theta) = \mathbf{a}(\theta) \otimes \mathbf{a}^*(\theta). \quad (11)$$

From [6], we know that the number of sources that the array can estimate is determined by the rank of $\mathbf{B}$. As in the case of MUSIC algorithm, we can compute the eigen decomposition of $\mathbf{C_X}$. Its eigenvectors $(\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_{M^2})$ are separated into the signal and noise subspaces according to the eigenvalues. Let $\mathbf{E}_1 = span\{\mathbf{u}_{D+1}, \dots, \mathbf{u}_{M^2}\}$ be the noise subspace; then, the "spatial spectrum" in the MUSIC-like estimator is defined as

$$\Psi(\theta) = \left\|\mathbf{E}_1^H \mathbf{b}(\theta)\right\|^{-2}. \quad (12)$$

The $D$ estimates of source directions can be obtained by searching the peaks of $\Psi(\theta)$.

An optimal array has its sensors located such that a maximum number of virtual sensors is obtained. In MUSIC-like estimator, this maximum is $M^2 - M + 1$ for $M$ physical sensors. For example, a Uniform Circular Array (UCA) of $M$ odd identical sensors is optimal [6].

## III. CAM Estimator

In this section, we propose the conjugate augmented MUSIC (CAM) estimator. For an array with $M$ physical sensors, the crosscorrelation functions $R_{x_m x_1}(\tau)$ between signal outputs $\{x_m(t)\}$ and $x_1(t)$ can be represented [8] as

$$\begin{aligned} &R_{x_m x_1}(\tau) \\ &\quad = E\left\{x_m\left(t+\frac{\tau}{2}\right)x_1^*\left(t-\frac{\tau}{2}\right)\right\} \\ &\quad = \sum_{i=1}^{D} R_{s_i s_i}(\tau) \\ &\qquad \times \exp\left[-\frac{j2\pi\left((a_m - a_1)\cos(\theta_i) + (b_m - b_1)\sin(\theta_i)\right)}{\lambda}\right] \\ &\quad + R_{n_m n_1}(\tau), \quad \tau > 0. \end{aligned} \quad (13)$$

The autocorrelation function $R_{s_i s_i}(\tau)$ in (13) can be evaluated as

$$\begin{aligned} R_{s_i s_i}(\tau) &= E\left\{s_i\left(t+\frac{\tau}{2}\right)s_i^*\left(t-\frac{\tau}{2}\right)\right\} \\ &= E\left\{A_i e^{j\omega_i\left(t+\frac{\tau}{2}\right)}A_i e^{-j\omega_i\left(t-\frac{\tau}{2}\right)}\right\} \\ &= A_i^2 e^{j\omega_i\tau} \end{aligned} \quad (14)$$

and is found to have same form as the source signals. The second term in (13) is evaluated as

$$\begin{aligned} R_{n_m n_1}(\tau) &= E\left\{n_m\left(t+\frac{\tau}{2}\right)n_1^*\left(t-\frac{\tau}{2}\right)\right\} \\ &= \sigma^2\delta(\tau)\delta(m-1) = 0. \end{aligned} \quad (15)$$

Noting that $(a_1, b_1) = (0, 0)$, (13) can be simplified to

$$R_{x_m x_1}(\tau) = \sum_{i=1}^{D} R_{s_i s_i}(\tau) \times \exp\left[-\frac{j2\pi\left(a_m\cos(\theta_i) + b_m\sin(\theta_i)\right)}{\lambda}\right]. \quad (16)$$

Therefore, similar to (1) and (2), (16) can be put into a vector form as

$$\mathbf{R_X}(\tau) = \mathbf{AR_S}(\tau) \quad (17)$$

$$\mathbf{R'_X}(\tau) = \mathbf{A'R_S}(\tau) \quad (18)$$

where

$$\begin{aligned}
\mathbf{R_X}(\tau) &= [R_{x_1x_1}(\tau), R_{x_2x_1}(\tau), \ldots, R_{x_Mx_1}(\tau)]^T \\
\mathbf{R'_X}(\tau) &= [R_{x_2x_1}(\tau), R_{x_3x_1}(\tau), \ldots, R_{x_Mx_1}(\tau)]^T \\
\mathbf{R_S}(\tau) &= [R_{s_1s_1}(\tau), R_{s_2s_2}(\tau), \ldots, R_{s_Ds_D}(\tau)]^T \\
\mathbf{A}' &= [\mathbf{a}'(\theta_1), \ldots, \mathbf{a}'(\theta_D)] \\
\mathbf{a}'(\theta_i) &= \begin{bmatrix} \exp\left(-j\frac{2\pi}{\lambda}\left(a_2\cos(\theta_i)+b_2\sin(\theta_i)\right)\right) \\ \exp\left(-j\frac{2\pi}{\lambda}\left(a_3\cos(\theta_i)+b_3\sin(\theta_i)\right)\right) \\ \vdots \\ \exp\left(-j\frac{2\pi}{\lambda}\left(a_M\cos(\theta_i)+b_M\sin(\theta_i)\right)\right) \end{bmatrix}
\end{aligned} \tag{19}$$

and $\mathbf{a}'(\theta_i)$ is the same as $\mathbf{a}(\theta_i)$, except that the first element is missing. Taking the conjugate of (18) and changing the argument to $-\tau$, we obtain

$$(\mathbf{R'_X}(-\tau))^* = (\mathbf{A}')^*\mathbf{R_S^*}(-\tau) = (\mathbf{A}')^*\mathbf{R_S}(\tau). \tag{20}$$

Combining (17) and (20), we obtain the conjugate augmented correlation vector $\mathbf{R}(\tau)$ as

$$\mathbf{R}(\tau) = \begin{bmatrix} \mathbf{R_X}(\tau) \\ (\mathbf{R'_X}(-\tau))^* \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ (\mathbf{A}')^* \end{bmatrix} \mathbf{R_S}(\tau). \tag{21}$$

Following the approach in [9], we form the pseudo-data matrix of $\mathbf{R}(\tau)$ for different lags

$$\overline{\mathbf{R}} = [\mathbf{R}(T_s), \mathbf{R}(2T_s), \ldots, \mathbf{R}(N_pT_s)] \tag{22}$$

where $T_s$ is the pseudo sampling period, and $N_p$ is the number of pseudo snapshots.

The corresponding FO cumulants matrix therefore becomes

$$\begin{aligned}
\mathbf{C}_{\overline{\mathbf{R}}} &= E\left\{(\overline{\mathbf{R}}\otimes\overline{\mathbf{R}}^*)(\overline{\mathbf{R}}\otimes\overline{\mathbf{R}}^*)^H\right\} - E\{\overline{\mathbf{R}}\otimes\overline{\mathbf{R}}^*\} \\
&\cdot E\left\{(\overline{\mathbf{R}}\otimes\overline{\mathbf{R}}^*)^H\right\} - E\{\overline{\mathbf{R}}\overline{\mathbf{R}}^H\}\otimes E\left\{(\overline{\mathbf{R}}\overline{\mathbf{R}}^H)^*\right\}.
\end{aligned} \tag{23}$$

Note that $\mathbf{C}_{\overline{\mathbf{R}}}$ is generated through two expectation operations. The first expectation is taken with respect to the signals $\{x_m(t)\}$ and $x_1(t)$. The second expectation is to obtain the FO cumulants of the set of pseudo-data vectors with different time lags.

Let $\overline{\mathbf{b}}(\theta_i)$ be

$$\begin{aligned}
\overline{\mathbf{b}}(\theta_i) &= \begin{bmatrix} \mathbf{a}(\theta_i) \\ (\mathbf{a}'(\theta_i))^* \end{bmatrix} \otimes \begin{bmatrix} \mathbf{a}(\theta_i) \\ (\mathbf{a}'(\theta_i))^* \end{bmatrix}^* \\
&= \begin{bmatrix} \mathbf{a}(\theta_i)\otimes\mathbf{a}^*(\theta_i) \\ \mathbf{a}(\theta_i)\otimes\mathbf{a}'(\theta_i) \\ (\mathbf{a}'(\theta_i))^*\otimes\mathbf{a}^*(\theta_i) \\ (\mathbf{a}'(\theta_i))^*\otimes\mathbf{a}'(\theta_i) \end{bmatrix} = \begin{bmatrix} \text{Group1} \\ \text{Group2} \\ \text{Group3} \\ \text{Group4} \end{bmatrix}
\end{aligned} \tag{24}$$

where the grouping of elements will be elaborated in the next section. Putting the set of $\{\overline{\mathbf{b}}(\theta_i)\}$ in matrix form, we obtain

$$\overline{\mathbf{B}} = \left[\overline{\mathbf{b}}(\theta_1), \ldots, \overline{\mathbf{b}}(\theta_D)\right]. \tag{25}$$

Similar to (8), after diagonalizing $\mathbf{C}_{\overline{\mathbf{R}}}$, we obtain

$$\mathbf{C}_{\overline{\mathbf{R}}} = \overline{\mathbf{B}}\overline{\mathbf{\Sigma}}_{\mathbf{S}}\overline{\mathbf{B}}^H \tag{26}$$

where $\overline{\mathbf{\Sigma}}_{\mathbf{S}}$ is the diagonal matrix of $\mathbf{R_S}(\tau)$.

Here, temporal information $\tau$ (delay) is used along with spatial information of the signal. As in the case of the MUSIC-like estimator, the eigenvectors denoted as $(\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{(2M-1)^2})$ can be separated into the signal and noise subspaces according to the eigenvalues. The "spatial spectrum" can be computed as in (12). The $D$ source directions can be identified from the maxima of the "spatial spectrum."

## IV. Estimation Capacity of CAM Estimator

The estimation capacity is defined as the number of signals that can be estimated. It is numerically equal to the rank of $\overline{\mathbf{B}}$. For CAM, we can show that the columns of $\overline{\mathbf{B}}$ (generated by CAM) are similar to that of $\mathbf{B}$ (generated by MUSIC-like), except they are longer. For ULA, we can obtain a Vandermonde matrix by permuting the rows of the steering matrix of (21). As mentioned in Section II, the columns of $\mathbf{B}$ are linearly independent. Therefore, the columns of $\overline{\mathbf{B}}$ are also linearly independent. Hence, we only need to find the number of different elements in any one column of $\overline{\mathbf{B}}$ to obtain the rank. Among the four groups of elements in (24), we see that Group4 is a subset of Group1 and can be ignored as it does not contribute any new elements. Further derivation requires the specification of the sensor locations. We consider two important cases here.

Case I: ULA

For $\mathbf{a}(\theta)$ (same for $\mathbf{a}^*(\theta)$), the largest index of the exponential functions in $\mathbf{a}(\theta)\otimes\mathbf{a}(\theta)$ is $(M-1)+(M-1)=2M-2$. Therefore, there are $2M-1$ different elements in $\mathbf{a}(\theta)\otimes\mathbf{a}(\theta)$, including the element "1" [10]. Comparing $\mathbf{a}(\theta)\otimes\mathbf{a}(\theta)$ to the Group2 elements, we see that one of the $\mathbf{a}(\theta)$ is replaced by $\mathbf{a}'(\theta)$, where the element "1" is missing. Therefore, there are $2M-2$ different elements in Group2. The same argument applies to Group3 elements. Comparing Group1 to Group2, we see that only the element "1" is new. The same is true when compared with Group 3. Therefore, for CAM using ULA, the rank of $\overline{\mathbf{B}}$ is $2(2M-2)+1=4M-3$.

Case II: Optimal array

In $\mathbf{a}(\theta)\otimes\mathbf{a}(\theta)$ (same for $\mathbf{a}^*(\theta)\otimes\mathbf{a}^*(\theta)$), the $[(k-1)M+l]$th element and the $[(l-1)M+k]$th element are both equal to $a_k(\theta)a_l(\theta)$. Hence, the number of different elements for terms with $k\neq l$ is $\binom{M}{2}$. For $k=l$, there are $M$ additional different elements. Hence, the total is $\binom{M}{2}+M=M(M+1)/2$ [10]. Similar to the argument in case I, the number of different elements in Group2 and Group3 are each $M(M+1)/2-1$. There are $M^2-M+1$ different elements in Group1 according to [4]. Out of that, $2M-2$ elements are redundant when compared with Group2 and Group3. Adding up the contributions from the three groups, the rank of $\overline{\mathbf{B}}$ for the optimal array is $2M^2-2M+1$. Fig. 2 shows the number of virtual sensors of CAM and MUSIC-like estimators for optimal arrays.

To summarize, the number of virtual sensors for the ULA is $4M-3$, and for the optimal array, it is $2M^2-2M+1$.

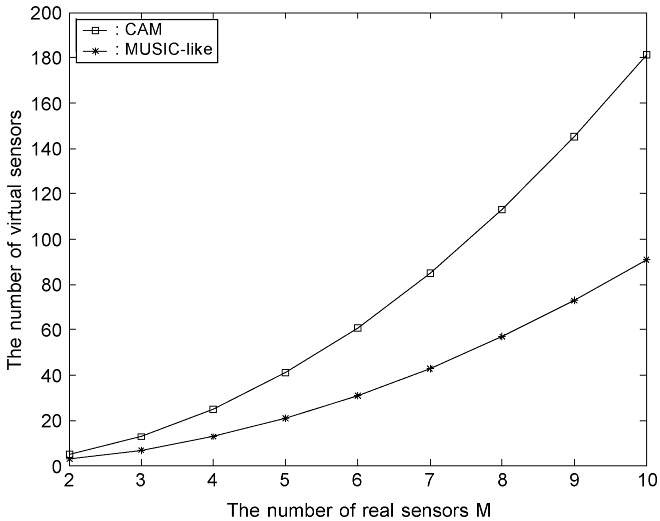


Fig. 2. Comparison of the number of virtual sensors that can be extended from an optimal array with $M$ physical sensors by different algorithms.

## V. CAM/ULA Estimator for ULA

The FO cumulants matrix $\mathbf{C}_{\overline{\mathbf{R}}}$ is a $(2M-1)^2\times(2M-1)^2$ matrix. To compute this matrix, for the arbitrary array, $(2M-1)^4$ operations are needed. In addition, a conventional eigendecomposition for an $\overline{M}\times\overline{M}$ matrix takes at least $o(\overline{M}^3)$ operations [11]. For ULA, this complexity can be significantly reduced, as shown below.

Let $\mathbf{A}''$ be the last row of matrix $\mathbf{A}$. From (16), the crosscorrelation function between the first and the $M$th sensor outputs is obtained as

$$R_{x_Mx_1}(\tau)=\mathbf{A}''\mathbf{R}_{\mathbf{S}}(\tau). \tag{27}$$

Taking the conjugate, and letting the argument be $-\tau$, we obtain

$$(R_{x_Mx_1}(-\tau))^*=(\mathbf{A}'')^*\mathbf{R}_{\mathbf{S}}(\tau). \tag{28}$$

Combining (27) and (28), a new correlation vector $\mathbf{R}''(\tau)$ is formed as

$$\mathbf{R}''(\tau)=\begin{bmatrix}R_{x_Mx_1}(\tau)\\(R_{x_Mx_1}(-\tau))^*\end{bmatrix}=\begin{bmatrix}\mathbf{A}''\\(\mathbf{A}'')^*\end{bmatrix}\mathbf{R}_{\mathbf{S}}(\tau). \tag{29}$$

Then, a pseudo-data matrix is obtained as

$$\overline{\mathbf{R}}''=[\mathbf{R}''(T_s),\mathbf{R}''(2T_s),\ldots,\mathbf{R}''(N_pT_s)]. \tag{30}$$

From (22) and (30), the corresponding FO cumulants matrix $\mathbf{C}''_{\mathbf{R}}$ can be computed as

$$\mathbf{C}''_{\mathbf{R}}=E\left\{(\overline{\mathbf{R}}''\otimes\overline{\mathbf{R}}^*)(\overline{\mathbf{R}}''\otimes\overline{\mathbf{R}}^*)^H\right\}-E\{\overline{\mathbf{R}}''\otimes\overline{\mathbf{R}}^*\}\cdot E\left\{(\overline{\mathbf{R}}''\otimes\overline{\mathbf{R}}^*)^H\right\}-E\{\overline{\mathbf{R}}''\overline{\mathbf{R}}''^H\}\otimes E\left\{(\overline{\mathbf{R}}\overline{\mathbf{R}}^H)^*\right\}. \tag{31}$$

After diagonalizing $\mathbf{C}''_{\mathbf{R}}$, we get

$$\mathbf{C}''_{\mathbf{R}}=\overline{\mathbf{B}}''\overline{\mathbf{\Sigma}}_{\mathbf{S}}\overline{\mathbf{B}}''^H \tag{32}$$

where

$$\overline{\mathbf{B}}''=\left[\overline{\mathbf{b}}''(\theta_1),\ldots,\overline{\mathbf{b}}''(\theta_D)\right]. \tag{33}$$

$$\overline{\mathbf{b}}''(\theta_i)=\begin{bmatrix}\exp\left(-j\frac{2\pi}{\lambda}(M-1)\cos(\theta_i)\right)\\\exp\left(j\frac{2\pi}{\lambda}(M-1)\cos(\theta_i)\right)\end{bmatrix}\otimes\begin{bmatrix}\mathbf{a}(\theta_i)\\(\mathbf{a}'(\theta_i))^*\end{bmatrix}^*. \tag{34}$$

Following the derivation in Section IV, the different elements in vector $\overline{\mathbf{b}}''(\theta_i)$ is found to be $4M-3$ (same as CAM for ULA). We call this estimator CAM/ULA. Since the dimension of $\mathbf{C}''_{\mathbf{R}}$ $((4M-2)\times(4M-2))$ is much small than that of $\mathbf{C}_{\overline{\mathbf{R}}}$, CAM/ULA can run much faster than CAM. A runtime comparison will be presented in Section VI. In addition, for ULA, the spatial function can be expressed in polynomial form, and the DOA estimates can be obtained from the roots of the polynomial rather than using the search procedure [12].

TABLE I
Average Runtime (in Seconds)

| $M$ | MUSIC-like | CAM | CAM/ULA |
|---|---|---|---|
| 3 | 0.11 | 0.83 | 0.56 |
| 4 | 0.12 | 1.25 | 0.78 |
| 5 | 0.20 | 1.96 | 0.87 |
| 6 | 0.28 | 2.96 | 1.06 |
| 7 | 0.35 | 4.80 | 1.37 |
| 8 | 0.50 | 8.53 | 1.58 |
| 9 | 0.68 | 12.22 | 1.86 |
| 10 | 1.05 | 18.50 | 2.23 |

## VI. Case Studies

We now evaluate the DOA estimation performance of CAM for a few cases by computer simulation. Specifically, the runtime of the algorithm, estimation capacity, angle resolution, sensitivity to noise, and the number of required snapshots of CAM are compared with that of MUSIC-like algorithm.

Consider a three-element uniform linear array with sensor separation $\lambda/2$ (this choice is to avoid any ambiguity in DOA estimation). The signals are assumed to be mutually independent and are of the form $s_i(t)=A_ie^{j\omega_it}$. For simplicity, let $A_i=A$ for all $i$. In the CAM algorithm, the pseudo sampling period $T_s$ is set to satisfy the sampling theorem. The SNR at each sensor is 10 dB. In addition, let $N_s$ and $N_p$ be the number of snapshots and pseudo snapshots, respectively.

### A. Case 1 (Runtime Comparison)

Case 1 is designed to compare the run time for the MUSIC-like, CAM, and CAM/ULA estimator. Table I records the average elapsed time of ten trial runs for various values of $M$. Here, $D=4$, $N_s=600$, and $N_p=300$. We use Matlab 6.1 to get the runtime on an AMD Athlon XP 1.8 GHz PC. Table I shows that for the same $M$, CAM has a much longer runtime than MUSIC-like, but the runtime of CAM/ULA appears to be of the same order of magnitude as MUSIC-like. The higher run time complexity of CAM is due to its higher estimation capacity.

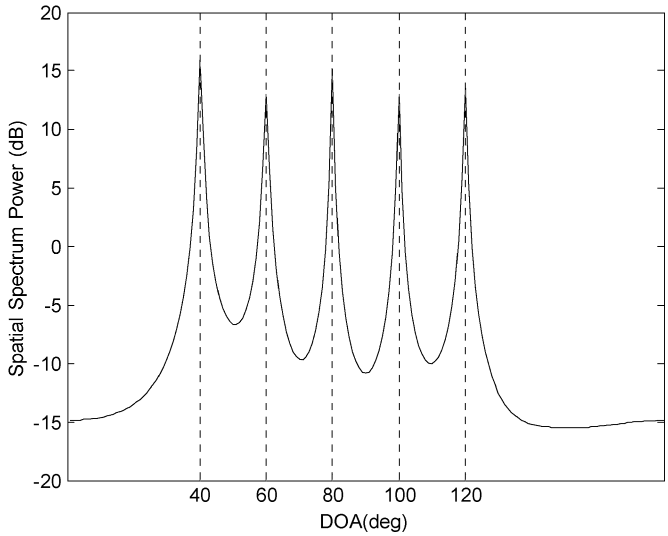


Fig. 3. Estimate five sources using the CAM algorithm. $N_s = 400$ and $N_p = 380$. SNR $= 10$ dB.

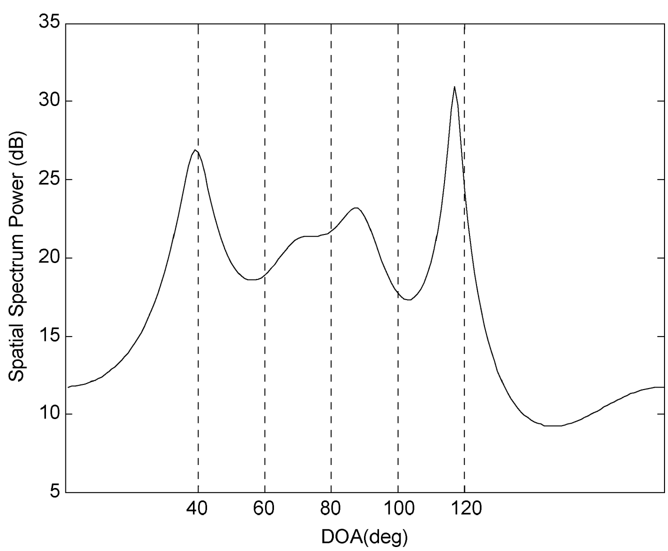


Fig. 4. Estimate five sources using the MUSIC-like algorithm. $N_s = 400$. SNR $= 10$ dB.

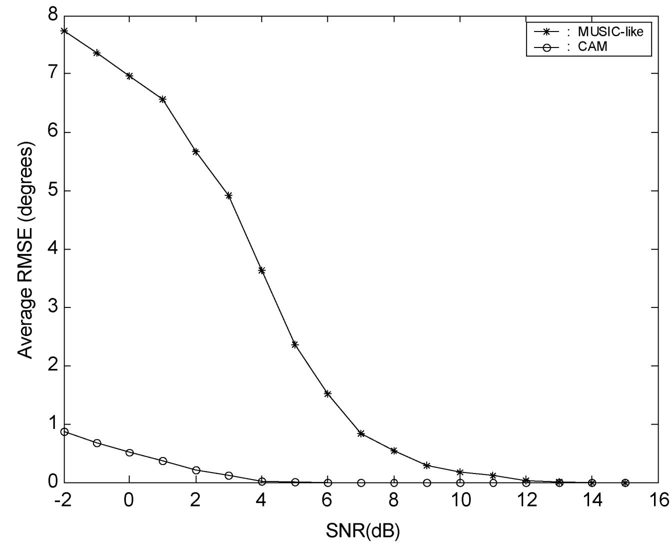


Fig. 5. Average RMSE of the MUSIC-like and CAM estimator versus SNR. Three element ULA, 200 trials, $N_s = 400$, and $N_p = 380$.

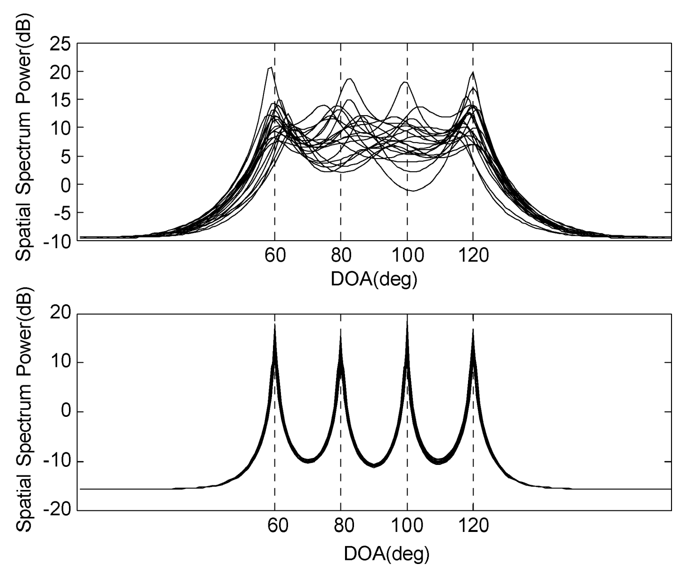


Fig. 6. Estimate four sources using MUSIC-like and CAM algorithms with SNR $= 0$ dB. Three element ULA, 20 trials, $N_s = 400$ and $N_p = 380$. MUSIC-like (upper) and CAM (lower).

## B. Case 2 (Estimation Capacity)

Suppose there were five sources impinging on the ULA with directions [40°, 60°, 80°, 100°, 120°]. Since virtual sensors can be extended from three physical sensors for the CAM estimator, these five source directions can be estimated correctly, as shown in Fig. 3. The use of the MUSIC-like estimator cannot obtain the correct source directions, as illustrated in Fig. 4.

## C. Case 3 (SNR Requirement Comparison)

In this case, we compare the performance of CAM and MUSIC-like in terms of SNR requirement. Let RMSE $= \sqrt{\sum_{i=1}^{D}(\theta_i - \hat{\theta}_i)^2}$ be the root-mean-square error of the $D$ direction estimations, where $\hat{\theta}_i$ is the estimate value of $\theta_i$. Let there be four signals coming from directions [60°, 80°, 100°, 120°]. Two hundred trail runs are carried out for estimators, and the average of 200 RMSE values are calculated and shown in Fig. 5 as a function of SNR. It is seen that at the same average RMSE value of 0.5°, CAM requires about 8 dB less SNR than MUSIC-like.

At SNR $= 0$ dB, the average RMSE values are 0.5° and 7° for CAM and MUSIC-like estimators, respectively. The results of 20 trial runs for both algorithms are shown in Fig. 6.

## D. Case 4 (Snapshot Requirement Comparison)

Fig. 6 shows that an average RMSE value of 0.5° is sufficient for estimating the directions for both MUSIC-like and CAM. We therefore use this value to get the required number of snapshots. Moreover, we let $N_p = N_s - 20$ and SNR $= 10$ dB. When four signals are impinged on a three-element ULA, we find that $N_s = 150$ and 200 are required for CAM and MUSIC-like, respectively, to make the average RMSE value less than 0.5°. Four signal directions are also the maximum that can be estimated by MUSIC-like. This shows that CAM does not require a greater number of snapshots for the same number of directions

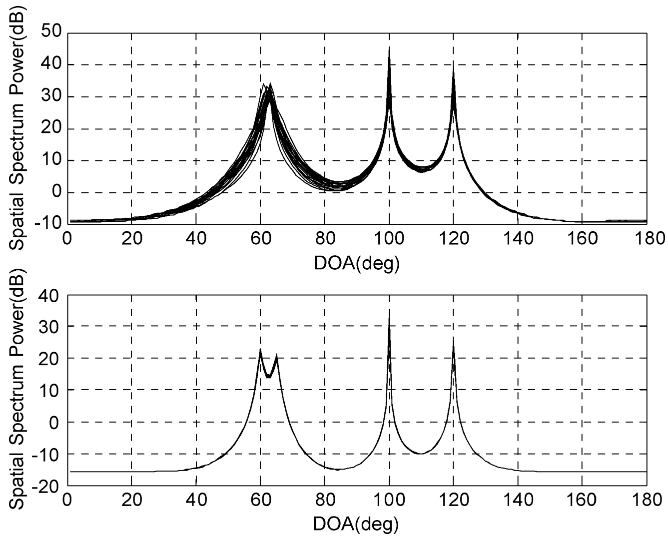


Fig. 7. Estimate four sources using the MUSIC-like (upper) and the CAM (lower) algorithms. Three element ULA, 20 trials, $N_s = 2000$, and $N_p = 1500$.

when compared with MUSIC-like. Using CAM, a maximum of eight signal directions can be estimated. The required number of snapshots is $N_s = 1700$.

### *E. Case 5 (Angle Resolution)*

In this simulation, we compare angle resolution for closely spaced sources. Let there be four sources with directions [60°, 65°, 100°, 120°] impinging on the same array mentioned before. Twenty trial runs are carried out, and the results are shown in Fig. 7. Here, $N_s = 2000$, and $N_p = 1500$. It is seen that the CAM estimator can resolve the two close-by directions (60° and 65°) whereas the MUSIC-like estimator presents difficulties.

### *F. Discussions*

The above cases show that the CAM estimator outperforms the MUSIC-like estimator in estimation capacity, SNR, snapshots, and angle resolution. These advantages are due to that fact that CAM estimator uses temporal information in addition to spatial information when estimating directions. Since CAM allows more virtual sensors be extended from physical sensors, the aperture of the CAM array is larger than that for MUSIC-like array. As an example, to obtain nine virtual sensors, $M = 3$ is needed for the CAM estimator, whereas $M = 5$ is needed for MUSIC-like estimator.

## VII. Conclusion

In this paper, we propose the CAM estimator for DOA estimation. It makes use of temporal information as well as spatial information when estimating directions. As a result, it can estimate a maximum of $2M(M-1)$ directions, which is twice that of the MUSIC-like estimator. When an ULA is used for CAM, the runtime can be significantly reduced. Computer simulation shows that CAM has higher estimation capacity, higher angle resolution, less sensitivity to noise, and requires fewer snapshots when compared with MUSIC-like.

## References

[1] H. Krim and M. Viberg, "Two decades of array signal processing research," *IEEE Signal Process. Mag.*, vol. 13, no. 7, pp. 67–94, Jul. 1996.
[2] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Signal Process.*, vol. ASSP-34, no. 2, pp. 276–280, Mar. 1986.
[3] S. V. Schell, R. A. Calabretta, W. A. Gardner, and B. G. Agee, "Cyclic MUSIC algorithms for signal-selective direction estimation," in *Proc. IEEE ICASSP*, Glasgow, U.K., May 1989, pp. 2278–2281.
[4] B. Porat and B. Friedlander, "Direction finding algorithms based on high-order statistics," *IEEE Trans. Signal Process.*, vol. 39, no. 9, pp. 2016–2023, Sep. 1991.
[5] M. C. Dogan and J. M. Mendel, "Application of cumulants to array processing-part I: aperture extension and array calibration," *IEEE Trans. Signal Process.*, vol. 43, no. 5, pp. 1200–1216, May 1995.
[6] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth-order direction finding problem," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2592–2595, Sep. 1999.
[7] A. N. Lemma, A.-J. van der Veen, and E. F. Deprettere, "Analysis of joint angle-frequency estimation using ESPRIT," *IEEE Trans. Signal Process.*, vol. 51, no. 5, pp. 1264–1283, May 2003.
[8] L. Jin and Q. Y. Yin, "Space-time DOA matrix method," *Acta Electronica Sinica*, vol. 28, pp. 8–12, June 2000.
[9] G. H. Xu and T. Kailath, "Direction-of-arrival estimation via exploitation of cyclostationarity—a combination of temporal and spatial processing," *IEEE Trans. Signal Process.*, vol. 40, no. 7, pp. 1775–1786, Jul. 1992.
[10] Q. Ding, P. Wei, and X. C. Xiao, "Estimation and analysis of DOA based on fourth-order cumulants," *Acta Electronica Sinica*, vol. 27, pp. 25–28, Mar. 1999.
[11] Y. H. Chen and Y. S. Lin, "Fourth-order cumulant matrices for DOA estimation," *Proc. Inst. Elect. Eng. Radar, Sonar Navig.*, vol. 141, pp. 144–148, Jun. 1994.
[12] A. J. Barabell, "Improving the resolution performance of eigenstructure-based direction-finding algorithms," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Aug. 1983, pp. 336–339.

**Zhilong Shan** received Ph.D. degree in electrical engineering from the South China University of Technology, Guangzhou, China, in 2004.

He is an Associate Professor with the School of Computers, South China Normal University. His research interests are sensor array processing and sensor networks.

**Tak-Shing P. Yum** (SM'86) was born in Shanghai, China. He received primary and secondary school education in Hong Kong. He received the B.S., M.S., M.Ph., and Ph.D. degrees from Columbia University, New York, NY, in 1974, 1975, 1977, and 1978, respectively.

He joined Bell Telephone Laboratories, Holmdel, NJ, in April 1978, working on switching and signaling systems. Two and a half years later, he accepted a teaching appointment at the National Chiao Tung University, Hsinchu, Taiwan, R.O.C. He stayed there for two years before joining The Chinese University of Hong Kong in 1982, where he is now Dean of engineering and Professor of information engineering. He has published widely in Internet research with contributions to routing, buffer management, deadlock handling, message resequencing, and multiaccess protocols. He has branched out to work on cellular networks, lightwave networks, and video distribution networks. His recent works are on 3G and IP networks. He enjoys doing research with students. Eight of his graduates are now professors at local and overseas universities. His diverse industrial experience includes Bell Labs, Bellcore (now Telcordia), IBM Research, Motorola Semiconductors and ITRI of Taiwan, SmarTone Communications, and Radio-Television (Hong Kong) Ltd. He has also lectured extensively in major universities in China and was appointed Adjunct Professor at South East University, Huazhong University of Science and Technology, and Zhejiang University.

Prof. Yum is on the editorial boards of six international journals on Communications and Information Science, including the IEEE Transactions on Communications. He was also formerly an editor of the IEEE Transactions on Multimedia.