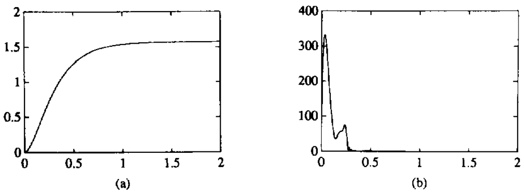

Fig. 2. Response when $\epsilon = 1.0$: (a) reference trajectory, (b) $||Y^T s||$.

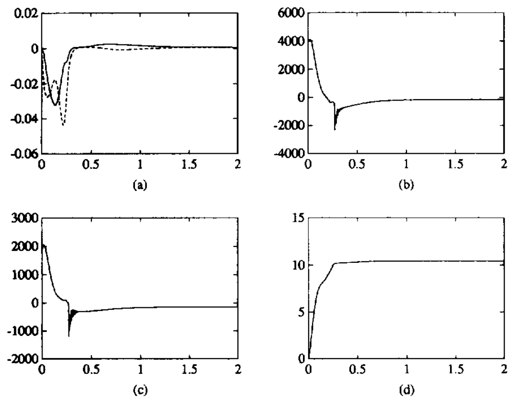

Fig. 3. Response when $\epsilon = 1.0$: (a) Tracking errors, (b) Torque input: joint 1, (c) Torque input: joint 2, (d) $\hat{\rho}(t)$.

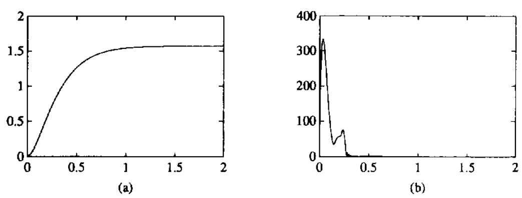

Fig. 4. Response when $\epsilon = 0.1$: (a) reference trajectory, (b) $||Y^T s||$.

are $\tilde{q}_1 = 7.6 \times 10^{-5}$ and $\tilde{q}_2 = 6.5 \times 10^{-5}$, and the steady-state estimate $\hat{\rho}$ is 10.47. The steady-state errors and estimate are smaller and larger than the previous ones, respectively because of the given small ϵ value. In this case, however, some chattering occurs at 0.8 seconds.

V. Conclusion and Discussion

In this note, we have presented a robust control law for robot manipulators based on Spong's work [1], in which a simple estimation law for uncertainty bound is proposed to exclude an assumption about the uncertainty bound. The proposed control scheme is easier to design than [1] because the pre-computation of the uncertainty bound is not needed. In practical application of adaptive control, its computational burden and persistency of excitation condition should be considered. In our scheme, since only one parameter, i.e., an upper bound on the uncertainty needs to be estimated using the proposed simple algorithm, it does not require much computation time and the persistent excitation condition.

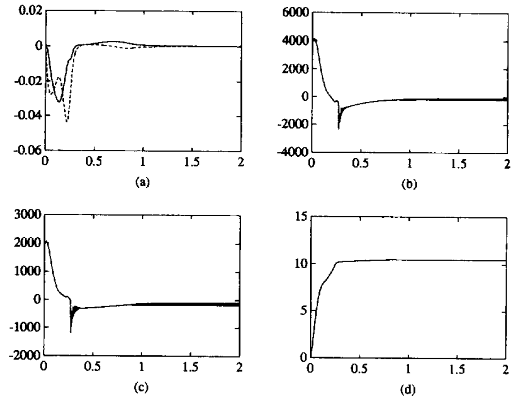

Fig. 5. Response when $\epsilon = 0.1$: (a) Tracking errors, (b) Torque input: joint 1, (c) Torque input: joint 2, (d) $\hat{\rho}(t)$.

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The Optimal Multicopy Aloha

Eric W. M. Wong and Tak-Shing P. Yum

***Abstract*—Multicopy Aloha is a generalization of single-copy Aloha where multiple copies of a packet are transmitted without first learning whether the transmission of the first copy is successful or not. Different users in the system can transmit different number of copies. But the optimal multicopy policy is found to be of the "pure" type whereby all users transmit the same number of copies. To maximize the probability of successful transmission we found that the optimal copy number is one for the normalized channel traffic Λ in range $0.48 < \Lambda \leq 1$, two for $0.28 < \Lambda \leq 0.48$, three for $0.20 < \Lambda \leq 0.28$, etc. Potential applications of multicopy Aloha include packet satellite systems, multichannel Aloha systems and reservation Aloha systems.**

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The authors are with the Department of Information Engineering, The Chinese University of Hong Kong, Hong Kong.

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I. Introduction

When m copies of a packet are transmitted on a slotted Aloha channel [1] one would be tempted to believe that the probability of successful transmission for that packet, or the probability that at least one of the m copies will not involve in a collision, would be higher than that when only one copy is transmitted. This is only true, however, when other packets are transmitting in single copies or when the channel traffic is very light because otherwise the higher probability of collision caused by the m times increase of traffic will more than offset the gain of diversity.

In this paper, we investigate the trade-off between these two factors and derive the optimal multiple-copy transmission policy under various channel traffic conditions. In addition, we compare two policies for multicopy Aloha:

1) Pure policy: All users transmit the same number of copies.
2) Mixed policy: Different users may transmit different number of copies.

To maximize the probability of successful transmission, we shall prove that the optimal pure policy is always better than the optimal mixed policy for Poisson arrivals. We choose to study only the slotted version as the unslotted version is just a minor variation.

Multicopy Aloha is not suitable for systems where the sensing of carriers is convenient. It is also not suitable for systems where the round-trip propagation delay between the transmitters and receivers is small because the kind of "look-ahead-retransmission" offered by multicopy Aloha is unwarranted in such systems. On the other hand, we do see potential application in the following three situations.

1) In packet satellite systems [2] with long round-trip propagation delay, multicopy Aloha can give higher probability of successful transmission than single-copy Aloha in light traffic conditions.
2) In an M-channel Aloha system [3], m copies of a packet can be distributed to m of the M available channels. For M large (necessary for the Poisson assumption stated in the next section to be valid), the same throughput gain can be obtained without randomly delaying the m copies as in the single channel case.
3) In a reservation Aloha system [2], the reservation sub-channel is in general very lightly loaded. The use of multicopy Aloha for the reservation packets can give a higher probability of successful reservation.

Multicopy Aloha is a generalization of the single copy Aloha. Its optimal use requires the knowledge of channel traffic. This knowledge, however, is always needed for channel stability control regardless of the copy numbers [2].

II. The Optimal Policy

Let the arrivals of packets be a Poisson process. For single copy slotted Aloha, it is generally assumed that if the average retransmission randomization delay is larger than five slots, the combined new and retransmitted packet arrivals can be approximated as a Poisson process [4]. In multicopy Aloha systems where packet copy transmissions are randomized in time, we make the analogous assumption that the combined new and retransmitted packet-copy transmissions constitute a Poisson process. The mean randomization delay for the m copies, including the first transmission, however, should be sufficiently large, say, no smaller than $5m$ slots.

Let us first divide the input arrivals into n types where a type i arrival ($i = 1, 2, \cdots, n$) consists of i copies of the same packet whose transmissions are randomized in time. Let λ_i be the rate of the type i arrivals (including retransmissions) and let $\Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ be the total packet arrival rate of all types. The average number of copies per packet N is given by

$$N = \Lambda^{-1} \sum_{i=1}^{n} i\lambda_i. \tag{1}$$

The probability P_i that a type i packet is successfully transmitted is

$$P_i = 1 - \text{Prob}\,[\text{all } i \text{ copies are collided}] = 1 - (1 - e^{-N\Lambda})^i.$$

The throughput contribution from type i packets, denoted as S_i, is $S_i = \lambda_i P_i$ and the total throughput S is

$$S = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \lambda_i [1 - (1 - e^{-N\Lambda})^i].$$

The probability of a successful transmission P for an arbitrary packet is just S/Λ and the average number of transmission attempts per packet T is given by Λ/S. For the m-copy Aloha ($m > 1$) with a mean randomization delay of $5m$ slots, the expected delay D for a packet to reach its destination [4] is

$$D = 5m + (R + 5m)(\Lambda/S - 1)$$

where R is the round-trip propagation delay on the channel. Therefore, for systems with a large R such as a packet satellite system, D is proportional to the average number of retransmissions $(\Lambda/S - 1)$. The multi-copy Aloha optimization problem can be formulated as maximizing S with respect to $\{\lambda_i\}$ such that $\sum_{i=1}^{n} \lambda_i = \Lambda$.

Let $S_k^{(pu)}$ be the throughput under the Pure Policy where all users transmit k copies per attempt. Then

$$S_k^{(pu)} = \Lambda P_k = \Lambda[1 - (1 - e^{-k\Lambda})^k] \qquad k = 1, 2, 3, \cdots. \tag{2}$$

Treat k in (2) as a continuous variable and differentiate (2) w.r.t. k, a global maximum at $k = (\ln 2)/\Lambda$ is found. (The second derivative is always negative.) Thus, if we choose Λ equal to $(\ln 2)/k$, where $k = 1, 2, 3, \cdots$, in (2), $S_k^{(pu)}$ is maximized to be

$$S_k^{(pu)}|_{\max} = \frac{\ln 2}{k}\left[1 - \left(\frac{1}{2}\right)^k\right] \qquad k = 1, 2, 3, \cdots.$$

The throughput under the mixed policy $S^{(mx)}$ is

$$S^{(mx)} = \sum_{i=1}^{n} \lambda_i [1 - (1 - e^{-N\Lambda})^i].$$

From Lemma 1 in the Appendix we obtain

$$S^{(mx)} < \Lambda[1 - (1 - e^{-N\Lambda})^N]. \tag{3}$$

At $\Lambda = (\ln 2)/k$ this upper bound is maximized at $N = (\ln 2)/\Lambda = k$. Thus using $N = k$ in (3), we have

$$S^{(mx)} < \frac{\ln 2}{k}\left[1 - \left(\frac{1}{2}\right)^k\right] = S_k^{(pu)}|_{\max} \tag{4}$$

at $\Lambda = (\ln 2)/k$. In other words, at these values of Λ, S is maximized with the use of the pure policy consisting of type k traffic only.

In the following, we present two theorems. Theorem 1 proves that even when $\Lambda \neq (\ln 2)/k$ the pure policy still gives higher throughput than the mixed policy. Theorem 2 gives the optimal pure policy for a given traffic rate Λ. The optimal number of copies k^* in the pure policy is found to increase with decreasing traffic load Λ as follows: $k^* = 1$ for $\Lambda \geq 0.48$, $k^* = 2$ for $0.28 \leq \Lambda < 0.48$, etc.

Let $\mathfrak{R}$ be the set of positive real numbers and let $\Omega = \mathfrak{R} - \{\ln 2, (\ln 2)/2, (\ln 2)/3, \cdots\}$.

Theorem 1: For $\Lambda \in \Omega$, S is maximized either when all users transmit the same $k^- = \lfloor \frac{\ln 2}{\Lambda} \rfloor$ copies per attempt or when all users transmit the same $k^+ = \lceil \frac{\ln 2}{\Lambda} \rceil$ copies per attempt.

Proof: First, consider the use of the pure policy. Since $S_k^{(pu)}$ has a global maximum at $k = (\ln 2)/\Lambda$, $S^{(pu)}$ is maximum at either

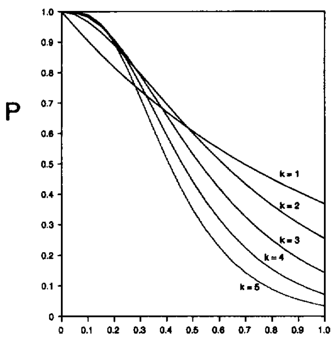

Fig. 1. P versus Λ.

$k^- = \lfloor (\ln 2)/\Lambda \rfloor$ or $k^+ = \lceil (\ln 2)/\Lambda \rceil$. It means that the maximum throughout of the pure policy occurs either when all users transmit the same k^- copies per attempt or when all users transmit the same k^+ copies per attempt.

Next, consider the use of the mixed policy. Depending on the value of N, we have three cases:

i. $N \le k^-$: From (3), the throughput of the mixed policy is bounded by $S^{(mx)} < \Lambda[1 - (1 - e^{-N\Lambda})^N]$. Since the bound has a single global maximum at $N = \ln 2/\Lambda$, for $N \le k^-$ the bound is obviously maximized at $N = k^-$, or $S^{(mx)} < \Lambda[1 - (1 - e^{-k^-\Lambda})^{k^-}] = S^{(pu)}_{k^-}$.

ii. $N \ge k^+$: Repeating the same argument in i), $S^{(mx)} < S^{(pu)}_{k^+}$ when $N \ge k^+$.

iii. $k^- < N < k^+$: Lemmas 2 and 3 in the Appendix stipulate that either the pure policy with $N = k^-$ or the pure policy with $N = k^+$ has throughput larger than that of any mixed policy with the same Λ. Q.E.D.

In the following, we shall, for convenience, suppress the superscript (pu) in the pure policy throughput notation.

Theorem 2: For each interval $\left[\frac{\ln 2}{k+1}, \frac{\ln 2}{k}\right]$ where $k = 1, 2, \cdots$, there exists a single point Λ_k of Λ given by the solution of

$$\frac{\ln(1 - e^{-k\Lambda})}{\ln(1 - e^{-(k+1)\Lambda})} = \frac{k+1}{k} \tag{5}$$

such that

$$\begin{aligned} S_{k+1}(\Lambda) &> S_k(\Lambda) && \text{for } \Lambda \in \left[\tfrac{\ln 2}{k+1}, \Lambda_k\right) \\ S_{k+1}(\Lambda) &= S_k(\Lambda) && \text{for } \Lambda = \Lambda_k \\ S_{k+1}(\Lambda) &< S_k(\Lambda) && \text{for } \Lambda \in \left(\Lambda_k, \tfrac{\ln 2}{k}\right]. \end{aligned}$$

Proof: From (2) it can be shown that

$$\begin{aligned} S_{k+1}(\Lambda) &> S_k(\Lambda) && \text{for } \Lambda = \tfrac{\ln 2}{k+1} \\ S_{k+1}(\Lambda) &< S_k(\Lambda) && \text{for } \Lambda = \tfrac{\ln 2}{k}. \end{aligned}$$

Since $S_k(\Lambda)$ and $S_{k+1}(\Lambda)$ are continuous functions of Λ, the curves of $S_k(\Lambda)$ and $S_{k+1}(\Lambda)$ against Λ must have one or more intersection points. Let Λ_k be the intersection point. Then Λ_k is the solution of $S_k(\Lambda) = S_{k+1}(\Lambda)$, or $\Lambda(1 - e^{-k\Lambda})^k = \Lambda(1 - e^{-(k+1)\Lambda})^{k+1}$. Canceling Λ, taking logarithm and rearranging, we obtain (5).

We now prove that the solution is unique. Let Δ be defined as

$$\Delta = (k+1)\ln(1 - e^{-(k+1)\Lambda}) - k\ln(1 - e^{-k\Lambda}).$$

To show that (5) has only one solution, it is sufficient to show

$$\frac{d\Delta}{d\Lambda} = \frac{(k+1)^2 e^{-(k+1)\Lambda}}{1 - e^{-(k+1)\Lambda}} - \frac{k^2 e^{-k\Lambda}}{1 - e^{-k\Lambda}} \tag{6}$$

is positive. Let us define a continuous function $F(x)$ as

$$F(x) = \frac{x^2 e^{-x\Lambda}}{1 - e^{-x\Lambda}}$$

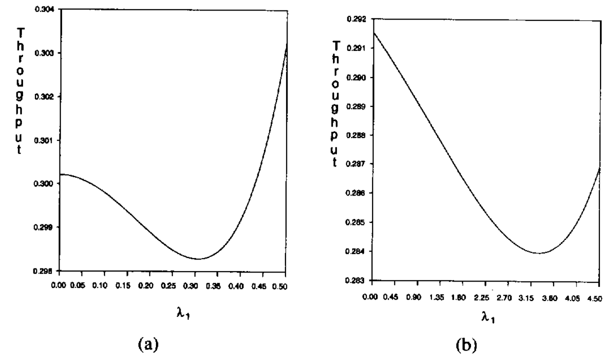

Fig. 2. Comparison of pure and mixed policies: (a) $\lambda_1 + \lambda_2 = 0.5$, (b) $\lambda_1 + \lambda_2 = 0.45$.

where $F(k+1) - F(k) = d\Delta/d\Lambda$. If we can prove that the derivative of $F(x)$ is positive for $k \le x \le k+1$, $F(k+1) - F(k)$ or $d\Delta/d\Lambda$ will indeed be positive. Differentiate $F(x)$, we have

$$\frac{dF(x)}{dx} = \frac{2x(2e^{x\Lambda} - 2 - x\Lambda e^{x\Lambda})}{(e^{x\Lambda} - 1)^2}. \tag{7}$$

Replacing all positive terms in (7) by their minimum bound values and all negative terms in (7) by their most negative bound values, we have

$$\frac{dF(x)}{dx} > \frac{2x\left[2^{\frac{2k+1}{k+1}} - 2 - \ln 2\left(\frac{k+1}{k}\right)2^{\frac{k+1}{k}}\right]}{(e^{x\Lambda} - 1)^2}. \tag{8}$$

It can be shown that $[\cdot]$ in (8) is positive for $k \ge 8$. Therefore, there is a unique Λ_k for $k \ge 8$. Numerical results show that for $k \le 7$ there is a unique solution for each k in the specified range. Q.E.D.

III. Numerical Results and Discussions

Fig. 1 shows P versus Λ with $1 \le k \le 5$. It shows that to maximize the probability of successful transmission, a single copy should be transmitted when the channel traffic Λ is larger than 0.48, when $\Lambda \in [0.28, 0.48]$, two copies should be transmitted, etc. It is seen that when Λ is small, P can be set arbitrarily close to one by using a large k.

Fig. 2 shows P versus λ_1 with $\Lambda = \lambda_1 + \lambda_2 = 0.5$ and with $\Lambda = \lambda_1 + \lambda_2 = 0.45$. It is seen that when $\Lambda = 0.5$ maximum S occurs at $\lambda_1 = 0.5$ or with the use of the pure policy with $k = 1$. When Λ decreases slightly to 0.45 (i.e., less than the $\Lambda = 0.48$ threshold) the maximum S location switches to $\lambda_1 = 0$ or $\lambda_2 = 0.5$. The optimal policy is the pure policy with $k = 2$. It is also seen that the throughput curves are not convex functions of λ_1. This causes some complications in proving Lemma 3.

Fig. 3 shows the optimal k as a function of Λ. It is seen that the optimal k decreases rapidly to one as Λ increases to 0.48. Classical single-copy Aloha analysis stipulates that S is maximized at $\Lambda = 1$. Thus for $0.48 < \Lambda \le 1$, the optimal k is one. Beyond $\Lambda = 1$, packets should be transmitted with probability $q = \Lambda^{-1}$ and blocked with probability $1 - q$ so as to maintain the effective traffic rate at one packet per slot.

Appendix

Lemma 1:

$$\Lambda[1 - (1 - e^{-\Lambda_c})^N] > \sum_{i=1}^{n} \lambda_i[1 - (1 - e^{-\Lambda_c})^i]. \tag{A1}$$

Proof: The inequality of the arithmetic and geometric means [5] states that

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} < \left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)^{p_1 + p_2 + \cdots + p_n}$$

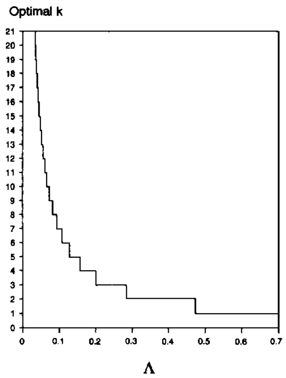

Fig. 3. Optimal multi-copy policy under different Λ.

where $a_i > 0$ and $p_i \geq 0$ with at least two p_i's not equal to zero. Let $p_i = \lambda_i$ and $a_i = (1 - e^{-N\Lambda})^i$, we obtain after simplification $\Lambda(1-e^{-N\Lambda})^N < \sum_{i=1}^{n} \lambda_i(1-e^{-N\Lambda})^i$. Adding Λ to both side and rearrange, we have

$$\Lambda - \Lambda(1 - e^{-N\Lambda})^N > \Lambda - \sum_{i=1}^{n} \lambda_i (1 - e^{-N\Lambda})^i.$$

Merging Λ into the summation, (A1) is obtained. Q.E.D.

Lemma 2: For nonintegral value of N, the throughput of the two-type policy consisting of type $\lfloor N \rfloor$ and type $\lceil N \rceil$ is larger than the throughput of any other mixed policy under the same Λ and N.

Proof: Let $m \equiv \lfloor N \rfloor$. Then $\lceil N \rceil = m + 1$. With type m and type $m+1$ in the system, we have

$$\lambda_m + \lambda_{m+1} = \Lambda$$

and

$$\frac{m\lambda_m + (m+1)\lambda_{m+1}}{\lambda_m + \lambda_{m+1}} = N.$$

The throughput under the above two-type policy is

$$S\mid_{\text{two-type}} = \lambda_m[1 - (1 - e^{-N\Lambda})^m] + \lambda_{m+1}[1 - (1 - e^{-N\Lambda})^{m+1}]. \tag{A2}$$

Consider any other mixed policy system with Λ and N given. Then, it is always possible to divide the traffic into two groups such that group 1 contains $\{\lambda_i^{(1)}\}$ and satisfies

$$\sum_{i=1}^{n} \lambda_i^{(1)} = \lambda_m \qquad \frac{\sum_{i=1}^{n} i\lambda_i^{(1)}}{\sum_{i=1}^{n} \lambda_i^{(1)}} = m$$

and group 2 contains $\{\lambda_i^{(2)}\}$ and satisfies

$$\sum_{i=1}^{n} \lambda_i^{(2)} = \lambda_{m+1} \qquad \frac{\sum_{i=1}^{n} i\lambda_i^{(2)}}{\sum_{i=1}^{n} \lambda_i^{(2)}} = m+1.$$

Using Lemma 1, the throughput contributions of the above two groups are bounded from above by

$$\lambda_m[1 - (1 - e^{-N\Lambda})^m] > \sum_{i=1}^{n} \lambda_i^{(1)}[1 - (1 - e^{-N\Lambda})^i]$$

$$\lambda_{m+1}[1 - (1 - e^{-N\Lambda})^{m+1}] > \sum_{i=1}^{n} \lambda_i^{(2)}[1 - (1 - e^{-N\Lambda})^i].$$

Adding up, we have $S_{\text{two-type}} > S^{(mx)}$, or the throughput of the two-type policy consisting of type $\lfloor N \rfloor$ and type $\lceil N \rceil$ is larger than that of any other mixed policy under the same Λ and N. Q.E.D.

Lemma 3: For noninteger values of $(\ln 2/\Lambda)$, the throughput of the pure policy with type $\lfloor (\ln 2/\Lambda) \rfloor$ or type $\lceil (\ln 2/\Lambda) \rceil$ traffic only is larger than the throughput of the two-type policy consisting of type $\lfloor (\ln 2/\Lambda) \rfloor$ and type $\lceil (\ln 2/\Lambda) \rceil$ traffic.

Proof: Let $m = \lfloor \frac{\ln 2}{\Lambda} \rfloor$ and $m + 1 = \lceil \frac{\ln 2}{\Lambda} \rceil$. Then from (A2), we have

$$S\mid_{\text{two-type}} = (\Lambda - \lambda_{m+1})[1 - (1 - e^{-m\Lambda - \lambda_{m+1}})^m] + \lambda_{m+1}[1 - (1 - e^{-m\Lambda - \lambda_{m+1}})^{m+1}]. \tag{A3}$$

Numerical results show that for $1 < (\ln 2/\Lambda) < 4$, S_2 is maximized either at $\lambda_{m+1} = 0$ or at $\lambda_{m+1} = \Lambda$. It means that for $1 < (\ln 2/\Lambda) < 4$, the throughput of the pure policy is always better than that of the two-type policy. For $(\ln 2/\Lambda) > 4$ (i.e., $m \geq 4$), we prove this analytically. First, from the definition of m, we have $(\ln 2)/(m+1) < \Lambda < (\ln 2)/m$. For $m \geq 4$, we have

$$\frac{4}{5}\ln 2 < m\Lambda < \ln 2. \tag{A4}$$

Therefore, for $m \geq 4$ we have

$$\lambda_{m+1} \leq \Lambda < (\ln 2)/m \leq (\ln 2)/4. \tag{A5}$$

Combining (A4) and (A5), we have

$$\frac{4}{5}\ln 2 < m\Lambda + \lambda_{m+1} < \frac{5}{4}\ln 2.$$

Therefore

$$\left(\frac{1}{2}\right)^{\frac{5}{4}} < e^{-m\Lambda - \lambda_{m+1}} < \left(\frac{1}{2}\right)^{\frac{4}{5}} \tag{A6}$$

$$1 - \left(\frac{1}{2}\right)^{\frac{4}{5}} < 1 - e^{-m\Lambda - \lambda_{m+1}} < 1 - \left(\frac{1}{2}\right)^{\frac{5}{4}}. \tag{A7}$$

Next, differentiate (A3) twice with respect to λ_{m+1} and let $G \equiv -m\Lambda - \lambda_{m+1}$, we have

$$\frac{dS_{\text{two-type}}}{d\lambda_{m+1}} = (1 - e^G)^{m+1} e^G [m\lambda_{m+1} e^G - m\Lambda + (1 - e^G)(1 - \lambda_{m+1})]$$

$$\begin{aligned}\frac{d^2 S_{\text{two-type}}}{d\lambda_{m+1}^2} = (1 - e^G)^{m-2} e^G \{ m e^G &\cdot [\lambda_{m+1}((m+2)e^G - 3) - m\Lambda + 2(1 - e^G)] \\ &+ [m\Lambda - (1 - e^G)^2(2 - \lambda_{m+1})]\}.\end{aligned} \tag{A8}$$

Replacing all positive terms in (A8) by their minimum bound values and all negative terms by their most negative bound values, i.e., using (A4)–(A7), we have, for $m \geq 4$

$$\begin{aligned}\frac{d^2 S_{\text{two-type}}}{d\lambda_{m+1}^2} > \Bigg\{ 4\left(\frac{1}{2}\right)^{\frac{5}{4}} \Bigg[\frac{\ln 2}{4}\left(6\left(\frac{1}{2}\right)^{\frac{5}{4}} - 3\right) &- \ln 2 + 2\left(1 - \left(\frac{1}{2}\right)^{\frac{4}{5}}\right)\Bigg] \\ &+ \left[\frac{4}{5}\ln 2 - 2\left(1 - \left(\frac{1}{2}\right)^{\frac{5}{4}}\right)^2\right]\Bigg\} > 0.\end{aligned}$$

This implies that $S_{\text{two-type}}$ is a convex $\cup$ function of λ_{m+1} for $m \geq 4$ and will attain its maximum at its boundary points $\lambda_{m+1} = 0$ or $\lambda_{m+1} = \Lambda$. Hence Lemma 3 is proved. Q.E.D.

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