Probabilistic Dynamics and Control of Linear Systems under Finite Communication Bandwidth Feedback†

Wing Shing Wong and Hui Cheng

***Abstract*—**In the paper, the asymptotic probabilistic behavior of systems stabilized by finite communication bandwidth feedback control is studied. The state orbits eventually converge to invariant intervals under the investigated finite bandwidth control laws. The optimal one-bit control laws that minimize either an asymptotic time-averaged expected cost function or an asymptotic expected cost function are derived. The asymptotic behavior of the closed-loop system is studied via invariant measures. The invariant distributions can be obtained from the induced matrix when the closed-loop system is a Markov transformation. It is shown that there are infinitely many cases of Markov transformations.

*Keywords***:** DFCB control, coded control system, invariant distribution, Markov transformation

† This work was supported by the Research Grants Council of Hong Kong under Grant 417105 and the National Nature Science Foundation of China under Grant 60804012.
W. S. Wong is with the Department of Information Engineering, the Chinese University of Hong Kong, Shatin, N.T., Hong Kong (phone: 852-26098378; fax: 852-26035032; e-mail: wswong@ie.cuhk.edu.hk;).
H. Cheng was with the Department of Information Engineering, the Chinese University of Hong Kong, Shatin, N.T., Hong Kong, and is now with the Department of Automation, Sun Yat-Sen University, China. (e-mail: chengh9@mail.sysu.edu.cn).

I. INTRODUCTION

LINEAR systems under finite bandwidth control have received much attention from a diverse group of researchers [1-19]. A common underlying premise in this research area is that the information between the plant and the controller is communicated over digital channels with finite capacity. Various control algorithms employing memoryless [1,3-9,13,14,19] or memory-based [10-12,15] coding have been proposed and analyzed, with emphasis on system stabilization. A recent survey paper [18] provides an overview of major research efforts in this field.

The overall objective of this paper is to understand the effect of controlling linear systems based on coded observations from a remote but accurate observer. For motivation, one can consider an earth-based vehicle which relies on a remote-sensing satellite for its coordinate measurements. The observation is optical based and can be assumed to be precise. The satellite informs the vehicle of its position via a digital communication channel with finite bandwidth. This paradigm is inherently different from a GPS system in which a vehicle can determine its coordinates from the triangulations of broadcast signals from multiple satellites. If the remote-sensing satellite has to support a large number of such vehicles simultaneously, the amount of communication bandwidth devoted to a single vehicle could be limited. As motivated by the concept of minimum attention control in [20], one can try to quantify the minimum amount of observation data rate needed for such a system to achieve a certain performance objective. One can also analysis the system performance at a given observation data rate.

More specifically, we focus on scalar linear dynamic systems in this paper. The system state, including the initial state, is unknown to the system but perfectly known to a remote observer. The initial system state however is distributed according to a known, compactly supported distribution. Performance of the system is defined as the asymptotic norm square of the state trajectory averaged over either the initial distribution or the infinite time horizon. Control laws considered in this system are time-invariant, memoryless, and based on finite code-words from a remote precise observer. Moreover, the control laws are assumed to be piecewise constant. It is known that the state trajectory of such a closed-loop feedback system behaves chaotically in general; as a result, the dynamic behavior of the feedback system is studied here via invariant measures [27]. In particular, the following results are established:

1. For systems employing single bit code-words at each sampling time and under the time-averaged performance index, the optimal law is derived.
2. For systems employing single bit code-words at each sampling time and under the asymptotic performance index, the optimal control law is derived if the closed-loop feedback system defines a covering map.

3. It is difficult to compute the invariant distribution in general. However, when the close-loop feedback system describes a Markov transformation [24] one can compute the invariant distribution by means of a finite-dimensional matrix. It is established in this paper that there are infinitely many of such Markov transformations relating to the single bit code-word control model.

Part of the work reported here bears resemblance to the work reported in [1]. There are, however, major differences. In [1], the objective is to understand the effect of quantized state measurements on a feedback control law. In this paper, we analyze feedbacks control law based on finite observation code-words with an aim to understand their asymptotic performance. It should also be pointed out that in [6], an invariant distribution for a particular multivariable system with specialized system parameters was computed explicitly. The approach here is built upon general invariant measure theory of maps defined on intervals and hence applies to a broad class of scalar systems. Although the model considered in the current paper does not include state and observation noises, the analysis presented here provides a glimpse of the complexity of performing detailed analysis of finite bandwidth control systems and could serve as a reference point for future work that incorporates the effect of various types of noise processes.

The rest of the paper is organized as follows: In Section II, stabilizability analysis of a feedback system under a single bit control law is discussed. In Section III, the concept of invariant measures is described and the optimal 1-bit control laws are derived. Computation of invariant distributions is studied in Section IV. Numerical results are provided in Section V and concluding remarks are presented in Section VI.

II. Feedback Systems with Finite Attention Bit Control Law

A. Problem Statement

Consider the plant that can be described by a scalar discrete-time system

$$\begin{cases} x_{n+1} = ax_n + u(z_n),\ x_0 \in \mathbb{R}, \\ z_n = \gamma(x_n), \end{cases} \tag{2.1}$$

where $x_n \in \mathbb{R}$ denotes the system state, $u(\cdot)$ represents the feedback control, and $a > 1$. The observation data are encoded into z_n by the function $\gamma(\cdot): \mathbb{R} \to \boldsymbol{S}$, where $\boldsymbol{S}$ denotes the codebook with a size 2^k. The data is then transmitted over a digital channel by a k-bit codeword at each time instance.

If the remote observer has to support a large number of systems, the feedback data rate for a single system could be limited. Hence, it is of interest to understand the controllability of systems using small sized code-words. In the extreme case, one can consider single bit codewords. Given an integer k, a natural question is to find the optimal coding scheme with $M = 2^k$ codes and a corresponding control law, u, that minimizes either an asymptotic averaged expected cost function

$$J_{ave} = \lim_{n\to\infty} \frac{1}{n} E\{\sum_{k=0}^{n-1} \|x_k\|^2\}, \tag{2.2}$$

or an asymptotic expected cost function

$$J_{\exp} = \lim_{n\to\infty} E\|x_k\|^2. \tag{2.3}$$

where the expectation is tanked with regard to the initial state pdf, f_0. These are known to be difficult problems in general.

Given a partition of the real line into $M \geq 2$ subintervals $\{R_i\}_{i=1}^M$ via the partition points, $-\infty < c_1 < \cdots < c_{M-1} < \infty$, consider a control function that is piecewise constant over the subintervals. Denote the control value in the subinterval $[c_{i-1}, c_i)$ by d_i, $i = 1, \ldots, M$. The dynamical evolution of (2.1) can be described in terms of a closed-loop system represented by the following piecewise affine map:

$$\upsilon(x) = \begin{cases} ax + d_1 & -\infty < x < c_1, \\ ax + d_2 & c_1 \leq x < c_2, \\ \vdots & \\ ax + d_M & c_{M-1} \leq x < \infty. \end{cases} \tag{2.4}$$

In this paper, we focus on the case where the inherent system is unstable, so that $a > 1$. Furthermore, we concentrate initially on feedback control law based on single bit code-words. For simplicity, we consider controllers that are essentially symmetric around the origin. So that the control functions are of the form:

$$u(x) = \begin{cases} -d & x \geq 0, \\ d & x < 0, \end{cases} \quad d > 0. \tag{2.5}$$

B. Stability of 1-Bit Control Laws

In order to ensure the cost function is finite, it is clear that the system should be containable or practically stable under the feedback control law [3]. Therefore, the following is a necessary and sufficient condition for the cost function to be bounded for the optimal control:

$$a \leq 2 \tag{2.6}$$

In the following, we study the asymptotic behavior of the proposed single bit feedback control law. The closed-loop system defined by such a controller can be represented by an expanding piecewise affine map

$$\tau(x)=\begin{cases} ax-d & x\geq 0, \\ ax+d & x<0, \end{cases} \quad a>1. \tag{2.7}$$

An interval I is said to be invariant under τ, if $x\in I$ implies $\tau(x)\in I$. The following Lemma shows the existence of invariant intervals for τ.

Lemma 2.1: For $1<a\leq 2$, the whole real line is partitioned into the three τ-invariant intervals, i.e., $(-\infty, d/(a-1))$, $[-d/(a-1), d/(a-1)]$, and $(d/(a-1),+\infty)$. In addition, the interval $\hat{J}=[-d,d]$ and the points $\{-d/(a-1), d/(a-1)\}$ are invariant with respect to τ.

The proof of this result is straightforward and is omitted.

Denote the interval $[-d/(a-1), d/(a-1)]$ by I^* and the set $\hat{J}\cup\{-d/(a-1), d/(a-1)\}$ by J^*. The following definition of stability from [11] is useful for subsequent discussions.

Definition 2.1 [11]: Given two intervals $J\subseteq I$, which are τ- invariant, τ is said to be (I,J)-stable if for every $x_0\in I$, there exists $n_0>0$ such that $x_n\in J$ for every $n\geq n_0$.

Lemma 2.2: For $1<a\leq 2$, τ is (I^*,J^*)-stable.

The proof of this result is straightforward and is omitted.

One can show that if $x\notin I^*$, then $\tau^n(x)\to\pm\infty$ as $n\to\infty$. Let $\mathrm{spt}f$ denote the support for a pdf f. If f_0 is a piecewise continuous distribution density and if $\mathrm{spt}f_0\not\subset I^*$, then there exists a small interval outside of I^* such that $|\tau^n(I^*)|\to\infty$ as $n\to\infty$. Hence, if the cost function is finite, then

$$\mathrm{spt}f_0\subseteq I^*. \tag{2.8}$$

III. Invariant Distributions and Optimal Single Bit Control Law

From now on assume that the initial distribution f_0 is continuous and has a piecewise continuous density function. To analyze the dynamics of an expanding piecewise affine transformation one can resort to the idea of invariant measures. Let $L^1(I,\mathcal{B},\lambda)$ be a normalized measure space with a Lebesgue measure λ on an interval I, $\mathcal{B}$ be the Borel field of I, and $\xi: I\to I$ be a measurable nonsingular map. The Frobenius-Perron operator $P_\xi: L^1\to L^1$ is defined as follows [21]: For any $f\in L^1$, $P_\xi f$ is the unique function in L^1 such that

$$\int_A P_\xi f\,d\lambda=\int_{\xi^{-1}(A)} f\,d\lambda \tag{3.1}$$

for any $A\in\mathcal{B}$. Denote the pdf corresponding to $\xi^n(x)$ by f_n, then

$$f_n = P_\xi^n f_0 . \tag{3.2}$$

A probability density f^* is ξ-invariant if and only if $P_\xi f^* = f^*$ almost everywhere. It is important to compute the invariant density function f^* since it describes the asymptotically dynamical behavior of the state orbit $\{\xi^n(x)\}_{n\geq 0}$. Moreover, resorting to the idea of invariant measures, the optimal single bit control laws that minimize the cost functions defined in equations (2.2) and (2.3) are derived, respectively.

Theorem 3.1: Given an initial distribution f_0 with compact support, the optimal control value d^* that minimizes the cost function $J_{ave} = \lim_{n\to\infty} \frac{1}{n} E\{\sum_{k=0}^{n-1} \|x_k\|^2\}$ under the map τ defined in (2.7) is

$$d^* = \arg\min_d \text{ such that } I^*(d) \supseteq \operatorname{spt} f_0 . \tag{3.3}$$

Proof: For the expanding piecewise linear transformation τ, it is shown in Theorem 1 [21] that $\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f_0 - f^* \right\| = 0$. Since τ and its derivative have only one discontinuity point, it follows from Theorem 1 [22] that τ has a unique invariant measure in L^1. Let $J_{ave}(d)$ and $f_d^*(x)$ denote the cost function and the invariant density related to the control value d, respectively. It can be shown that the invariant density functions $f_1^*(x)$ and $f_d^*(x)$ are related as follows:

$$f_d^*(x) = (1/d) f_1^*(x/d), \ x \in \mathbb{R} .$$

If the control value d is set so that $\operatorname{spt} f_0 \cap ((-\infty, -d/(a-1)) \cup (d/(a-1), \infty)) \neq \varnothing$, then by Lemma 2.2, J_{ave} is unbound. Hence to optimize J_{ave}, one has to set $[-d/(a-1), d/(a-1)] \supseteq \operatorname{spt} f_0$. We have

$$\begin{aligned} J_{ave}(d) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} E\{\|x_k\|^2\} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} x^2 P_\tau^k f_0 \, dx \\ &= \int_{-d}^{d} x^2 (\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f_0) \, dx = \int_{-d}^{d} x^2 f_d^*(x) dx = \int_{-d}^{d} x^2 [(1/d) f_1^*(x/d)] dx \\ &= d^2 \int_{-1}^{1} x^2 f_1^*(x) dx = d^2 \int_{-1}^{1} x^2 (\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f_0) \, dx = d^2 J_{ave}(1). \end{aligned} \tag{3.4}$$

This implies that $J_{ave}(d)$ can be minimized by choosing d to be the smallest value satisfying the condition $[-d/(a-1), d/(a-1)] \supseteq \operatorname{spt} f_0$. □

For the asymptotic expected cost function $J_{\exp} = \lim_{n\to\infty} E\|x_k\|^2$, results similar to Theorem 3.1 can be obtained when the map τ is a covering. The concept of a *covering map* is defined as follows.

Definition 3.1 [9]: a map $\xi : I \to I$ is said to be a covering, if for any open interval $U \subseteq I$, there exists $t \in \mathbb{N}$ such that $\xi^t(U) = I$.

The following definition introduces the concept of *asymptotical stability* of a Frobenius-Perron operator.

Definition 3.2 [26]: Let P_ξ be a Frobenius-Perron operator corresponding to a nonsingular transformation $\xi : I \to I$. Then $\{P_\xi^n\}$ is said to be asymptotically stable if there exists a unique density function f^* such that $P_\xi f^* = f^*$ and $\lim_{n\to\infty} \left\| P_\xi^n f - f^* \right\| = 0$ for any $f \in L^1$.

As stated in Lemma 2.1, τ maps $\hat{J} = [-d, d]$ to itself. By a slight abuse of notation, one can denote the map restricted to $\hat{J}$ as τ.

Theorem 3.2: Given an initial distribution f_0 with compact support, if τ restricted to $\hat{J}$ is a covering, the optimal control value d^* that minimizes the cost function $J_{\exp} = \lim_{n\to\infty} E\left\| x_n \right\|^2$ is

$$d^* = \arg\min_d \text{ such that } I^*(d) \supseteq \operatorname{spt} f_0 . \tag{3.5}$$

Proof: Lemma 2.2 shows that τ is (I^*, J^*)-stable when $1 < a \le 2$. Note that all trajectories starting in I^* ends in $\hat{J}$ with the exception of $-d/(a-1)$ and $d/(a-1)$ which are fixed points of τ. Let $\mathcal{P} = \{I_i\}_{i=1}^q$ be a partition of the interval $\hat{J} = [-d, d]$. For the covering map τ, denote m the smallest integer such that $\tau^m(I_i) = \hat{J}$, $i = 1, \ldots, q$. As shown in Theorem 6.2.2 [26], the map $P_{\tau^m}(x)$ is asymptotically stable:

$$\lim_{k\to\infty} \left\| P_{\tau^m}^k f_0 - f^* \right\| = \lim_{k\to\infty} \left\| P_\tau^{mk} f_0 - f^* \right\| = 0 . \tag{3.6}$$

For any $0 \le l < m$ this implies,

$$\lim_{k\to\infty} \left\| P_\tau^{mk} (P_\tau^l f_0) - f^* \right\| = 0 . \tag{3.7}$$

For an integer n in the form $n = mk + l$, this implies that $\{P_\tau^n\}$ is asymptotically stable, that is,

$$\lim_{n\to\infty} \left\| P_\tau^n f_0 - f^* \right\| = 0 . \tag{3.8}$$

Following this convergence result via the arguments in Theorem 3.1, we have

$$\begin{aligned} J_{\exp}(d) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} x^2 P_\tau^n f_0 \, dx = \lim_{n\to\infty} \int_{-d}^{d} x^2 P_\tau^n f_0 \, dx = \int_{-d}^{d} x^2 f_d^*(x) \, dx \\ &= \int_{-d}^{d} x^2 [(1/d) f_1^*(x/d)] dx = d^2 \int_{-1}^{1} x^2 f_1^*(x) dx = d^2 \int_{-\infty}^{\infty} x^2 f_1^*(x) dx = d^2 J_{\exp}(1). \end{aligned} \tag{3.9}$$

This implies that $J_{\exp}$ can be minimized by choosing d to be the smallest value satisfying the condition $[-d/(a-1), d/(a-1)] \supseteq \operatorname{spt} f_0$. □

Given there is a condition on τ being a covering in Theorem 3.2, it is natural to ask how restrictive such a condition is. Proposition 3.1 partially addresses this question and is a slight modification of a theorem of Williams [27], adapted to the system considered here.

Proposition 3.2: If $a > \sqrt{2}$, the map τ is a covering over the interval $\hat{J}$.

Proof: Define an open interval $U \subseteq \hat{J}$. Using the proof of Williams Theorem [27], one can show that there exists an integer k so that the interval $\tau^k(U)$ finally contains either $[0,d)$ or $(-d,0]$. Assume that $\tau^k(U) \supseteq [0,d)$. This implies

$$\begin{aligned} &\tau^{k+1}(U) \supseteq [-d, \tau(d)) = [-d,0) \cup [0, \tau(d)), \text{ and} \\ &\tau^{k+2}(U) \supseteq [-d, \tau^2(d)) \cup [\tau(-d), d). \end{aligned} \tag{3.10}$$

The last union set is equal to $[-d,d)$ if $\tau^2(d) - \tau(-d) = a^2 - 2 > 0$. This holds as $a > \sqrt{2}$. Similar arguments hold for the case $\tau^k(U) \supseteq (-d,0]$. □

Note that Proposition 3.2 does not need to assume $\tau^2(d) > 0$ as stated in condition (iii) of William Theorem [27].

IV. Computation of the Invariant Distributions of Markov Transformations

In the preceding section, the optimal control values for single bit memoryless controllers have been characterized. To determine the value of the optimal cost achieved, one obvious approach is to first compute the invariant distribution for the corresponding closed-loop system, τ, defined in (2.7). General speaking, such a computing task is difficult. However, the computation can be greatly simplified if τ is a Markov transformation, which is defined as follows:

Definition 4.1 [24]: Let $I = [c,e]$ and $\xi : I \to I$ be a piecewise linear map and let $\mathscr{P} = \{I_i\}_{i=1}^q$ be a finite partition of I by the points $c = c_0 < c_1 < \cdots < c_q = e$. For $i = 1,\ldots,q$, let $I_i = (c_{i-1}, c_i)$ and denote the restriction of ξ to I_i by ξ_i. If ξ_i is a homeomorphism from I_i onto some connected union of intervals of $\mathscr{P}$, then ξ is said to be a Markov transformation, and $\mathscr{P}$ is a Markov partition under ξ.

Theorem 9.4.1 of [25] establishes that for an expanding piecewise linear Markov transformation, ξ, every ξ - invariant density function must be piecewise constant over the partition $\mathscr{P}$. A function that is piecewise constant over a partition of I can be represented by a row vector made up of the corresponding constant values. Following the approach in [25], given a Markov partition $\mathscr{P}$ defined by ξ and a function f that is piecewise constant over $\mathscr{P}$, the action of the Frobenius-Perron operator, P_ξ, on f can be represented by a $q \times q$ matrix M_ξ so that if $\pi^f = (\pi_1, \dots \pi_q)$ represents f, then $P_\xi f$ is represented by $\pi^f M_\xi$. The element of the matrix, $M_\xi = (m_{ij})_{1 \le i,j \le q}$, is defined by [25]:

$$m_{ij} = a_{ij} / |\xi'_i| = \lambda(I_i \cap \xi^{-1}(I_j)) / \lambda(I_i), \quad 1 \le i, j \le q, \tag{4.1}$$

where λ denote the Lebesgue measure. It can be shown that the induced matrix M_ξ has the following properties (Theorem 9.3.1 [25]): M_ξ has 1 as the eigenvalue of maximum modulus; if M_ξ is also irreducible, then the multiplicity of the eigenvalue 1 is 1. Therefore, for a map, τ, corresponding to a Markov transformation with an irreducible M_ξ, the invariant distribution can be uniquely determined via the eigenvector corresponding to the eigenvalue 1. Given this property, it is of interest to study conditions that guarantee τ is a Markov transformation. It is also of interest to know how common are such Markov transformations as one varies parameter a in system (2.7). In this section it is shown that there are indeed infinitely many such τ's.

For each $n \ge 1$, the partition $\mathscr{P}^n$ under the map τ is defined as follows [28]:

$$\mathscr{P}^n = \bigvee_{k=0}^{n-1} \tau^{-k}(\mathscr{P}) \triangleq \{I_{i_0} \cap \tau^{-1}(I_{i_1}) \cap \dots \cap \tau^{-n+1}(I_{i_n}) : I_{i_j} \in \mathscr{P}\}. \tag{4.2}$$

Let $\mathscr{Q}$ denote the set of partition points of $\mathscr{P}$, and define $\mathscr{Q}^n$ as $\mathscr{Q}^n = \bigcup_{k=0}^{n-1} \tau^{-k}(\mathscr{Q})$. It can be shown that $\mathscr{Q}^n$ is the set of partition points of $\mathscr{P}^n$. The endpoints of a Markov partition must be invariant under τ. The following definition of a Markov partition is adopted from [28].

Definition 4.2 [28]: A partition $\mathscr{P} = \{I_i\}_{i=1}^q$ is a Markov partition of order $r > 1$ if $\tau(\mathscr{Q}^r) \subseteq \mathscr{Q}^r$ and $\tau(\mathscr{Q}^{r-1}) \setminus \mathscr{Q}^{r-1} \ne \varnothing$.

One method of generating a Markov partition of order $r > 1$ is to start with a simple initial partition and investigate the action of τ^i on it. For the initial partition $\mathscr{P} = \{[-d, 0), [0, d]\}$, it was shown in [28] that $\mathscr{P}$ is a Markov partition of order $r > 1$ if and only if r is the smallest integer such that

$$\tau^r(\pm d) = 0. \tag{4.3}$$

From this observation, it was proven in [28] that a necessary and sufficient condition for τ to be a Markov transformation of order $r > 1$ is that the parameter a is a root to a polynomial of the form

$$(-1)^m a^{s_m+1} - 2\sum_{i=1}^{m}(-1)^i a^{s_i} - 1 = 0, \tag{4.4}$$

where $s_0 = 0$, $s_i = s_{i-1} + v_i$, $i = 1,...,m$, and $\mathbf{v}$ is a vector of positive integers, $\mathbf{v} = [v_1, v_2,..., v_m]$, with $v_i \le v_m$, such that m satisfies $\sum_{i=1}^{m} v_i = r$. (The last equation has some slight discrepancy with the original formulation in [28].) Although it has been pointed out in [28] that there can be at most countably infinite solution to (4.4), not much is known about the nature of these solutions.

In this paper, a constructive proof is presented identifying infinite families of Markov transformations, where each family contains infinitely many cases of Markov transformations, all contained within the range $1 < a < 2$. To facilitate the discussion, a few basic lemmas are required.

For $i \ge 1$, define the polynomial $f_i(x) = x^{i+1} - 2x^i + 1$. It follows that $f_i'(x) = (i+1)x^i - 2ix^{i-1}$. It is clear that any $f_i(x)$ has at least one real root, $x = 1$.

Lemma 4.1: For $i \ge 2$, the largest real root α_i of $f_i(x)$ satisfies the conditions:

1. $1 < \alpha_{i-1} < \alpha_i < 2$.

2. $\alpha_i^i > i$. (4.5)

3. For $x \ge \alpha_i$, $f_i'(x) > 0$, and $f_i(x) > 0$ for $x > \alpha_i$.

Proof: See Appendix A.

Let $g_i(x) = \dfrac{f_i(x)}{x-1} = x^i - \dfrac{x^i - 1}{x-1}$. It is clear that α_i is a real root of $g_i(x)$. The following result is needed for subsequent discussion.

Lemma 4.2: Suppose $i \ge 2$. For $x > \alpha_i$, $g_i(x) > 0$. For $x \ge \alpha_i$, $g_i'(x) > 0$.

Proof: See Appendix B.

Consider now the map $\tau_a^i(x)$ restricted to $[-d, d]$.

Lemma 4.3: Suppose $d > 0$ and $\tau_a^j(d) > 0$ for $j = 1,\ldots,i-1$, then

$$\tau_a^i(d) = [a^i - (a^{i-1} + a^{i-2} + \ldots + 1)]d = g_i(a)d. \tag{4.6}$$

The proof is straightforward and is omitted.

Proposition 4.1: For $a = \alpha_i$, $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_\infty = 2$, the close-loop maps, τ_a restricted to $[-d, d]$ are Markov transformations.

Proof: For $i = 1$, the claim is obvious. For $i > 1$, one can show by induction by using Lemma 4.1, 4.2, and 4.3 that

$$\tau_{\alpha_i}^i(d) = [\alpha_i^i - (\alpha_i^{i-1} + \alpha_i^{i-2} + \ldots + 1)]d = g_i(\alpha_i)d = 0\,. \tag{4.7}$$

Hence, an infinite family of Markov transformations can be constructed. The inequality

$$1 = \alpha_1 < \alpha_2 < \cdots < \alpha_\infty = 2 \tag{4.8}$$

follows from Lemma 4.1. □

Fig. 1 illustrates three cases of τ_{α_i}. Fig. 1(a) shows the case of $\alpha_1 = 1$. Fig. 1(b) shows the case of $\alpha_2 = (1+\sqrt{5})/2$. The fact that these define Markov transformations is well-known. Fig. 1(c) depicts a general case of τ_{α_i} for α_i when $i \geq 3$.

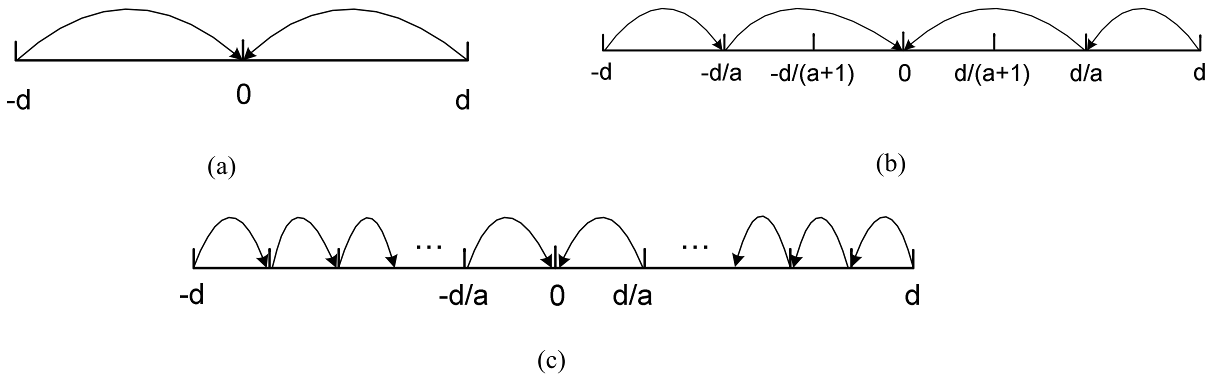

Fig. 1. Orbits of $\tau_{\alpha_i}(d)$: (a) $\alpha_1 = 1$; (b) $\alpha_2 = (1+\sqrt{5})/2$; (c) α_i for $i > 3$.

The set of Markov transformations is of course much more complex than the set $\{\alpha_1, \ldots, \alpha_\infty\}$. In fact, it will be shown that in the interval (α_i, α_{i+1}), for any $i \geq 1$, there are infinite many cases of Markov transformations. To facilitate subsequent discussions, define a sign sequence for an orbit as follows. Let

$$\operatorname{sgn}(x) = \begin{cases} + & \text{if } x > 0, \\ - & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{4.9}$$

The sign sequence for the orbit of x under τ_a is defined as:

$$\left\{\operatorname{sgn}(x), \operatorname{sgn}\left(\tau_a(x)\right), \operatorname{sgn}\left(\tau_a^2(x)\right), \ldots\right\}. \tag{4.10}$$

Remark 4.1: It has been pointed out $\{-d/(a+1), d/(a+1)\}$ are fixed points under τ_a. It is easy to verify the following result.

Lemma 4.4: (i) $\tau_a([0, d/(a+1)) = [-d, -d/(a+1)]$ and $\tau_a([-d/a, -d/(a+1)]) = [0, d/(a+1)]$.

(ii) For $0 \le x < d/(a+1)$, $\tau_a(x) < -d/(a+1)$, $\tau_a^2(x) < x$.

(iii) If $0 \le x < d/(a+1) - \varepsilon$, $\varepsilon \ge 0$, then $\tau_a^2(x) = d/(a+1) - a^2\varepsilon$.

Proof: See Appendix C.

Lemma 4.5: For any $i \ge 1$, there exists a unique β_i, such that $\alpha_i < \beta_i < \alpha_{i+1}$ and

$$g_i(\beta_i) = 1/(\beta_i + 1). \tag{4.11}$$

Moreover, $g_i(x) > 1/(\beta_i + 1)$ for $x \in (\beta_i, \alpha_{i+1}]$, $g_i(x) < 1/(\beta_i + 1)$ for $x \in (\alpha_i, \beta_i)$.

Proof: See Appendix D.

The point $d/(\beta_i + 1)$ has the special property that it is mapped to $-d/(\beta_i + 1)$ by τ_{β_i}; both points are fixed under $\tau_{\beta_i}^2$.

Proposition 4.2: For any integer $i \ge 2$, there exists an $\omega(i,1)$, $\omega(i,1) \in (\alpha_i, \beta_i)$, such that the sign sequence for the orbit of d under $\tau_{\omega(i,1)}$ is

$$\{\underbrace{+,+,+,\ldots,+,+}_{i+1},-,0,\ldots\}. \tag{4.12}$$

In other words, $\tau_{\omega(i,1)}^{i+2}(d) = 0$. Moreover, $\tau_x^{i+2}(d) > 0$ for $x \in (\omega(i,1), \beta_i]$.

For any integers $j \ge 2$, $i \ge 1$, there exists an $\omega(i,j)$, $\alpha_i < \omega(i,j-1) < \omega(i,j) < \beta_i$, such that the sign sequence for the orbit of d under $\tau_{\omega(i,j)}$ is

$$\{\underbrace{+,+,+,\ldots,+,+}_{i+1},\underbrace{-,+,\ldots,-,+,-,0}_{2j},\ldots\}. \tag{4.13}$$

In other words, $\tau_{\omega(i,j)}^{i+2j}(d) = 0$. Moreover, $\tau_x^{i+2j}(d) > 0$ for $x \in (\omega(i,j), \beta_i]$.

Proof: See Appendix E.

When $x \in (\beta_i, \alpha_{i+1}]$, the inequality in Lemma 4.4 (ii) is reversed in the sense that

$$\tau_a^2(x) > x. \tag{4.14}$$

In other words, $\tau_{\zeta(i,j)}^{i+2j}(d) = 0$. Moreover, $\tau_{\beta_i}^{i+2}(d) = d/(\beta_i + 1)$ and for x arbitrarily close to α_{i+1}, $\tau_{\alpha_{i+1}}^{i+2}(d)$ is arbitrarily close to d, so using arguments shown previously, there exists an $x \in (\beta_i, \alpha_{i+1})$ so that $\tau_x^{i+2}(d) = d/x$ and $\tau_x^{i+3}(d) = 0$. In general, one can establish the following result:

Proposition 4.3: For any integers $i \ge 1$ and $j \ge 1$, there exists an $\zeta(i,j)$, $\beta_i < \zeta(i,j+1) < \zeta(i,j) < \alpha_{i+1}$, so that the sign sequence for the orbit of d under $\tau_{\zeta(i,j)}$ is

$$\{\underbrace{+,+,+,\ldots,+,+}_{i+1},\underbrace{-,+,\ldots,-,+,-,+,0}_{2j-1},\ldots\}. \tag{4.15}$$

In other words, $\tau_{\zeta(i,j)}^{i+2j}(d)=0$. Moreover, $\tau_x^{i+2j}(d)<0$ for $x\in[\beta_i,\zeta(i,j))$.

The proof is similar to that of Proposition 4.2 and the details are omitted.

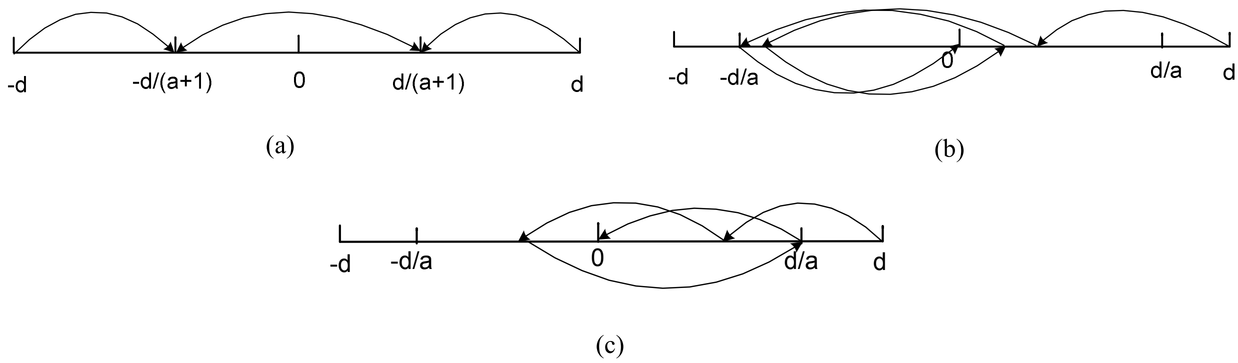

Fig. 2. Orbits of $\tau_x(d)$ for value of x near $\sqrt{2}$: (a) $\beta_1=\sqrt{2}$; (b) $\alpha_1<\omega(1,2)<\beta_1$; (c) $\beta_1<\zeta(1,2)<\alpha_2$.

Fig. 2 illustrates the behavior of $\tau_{\omega(1,2)}$ and $\tau_{\zeta(1,2)}$. Fig. 2(a) shows the case of $\beta_1=\sqrt{2}$. Fig. 2(b) shows the case of $\alpha_1<\omega(1,2)<\beta_1$. Fig. 2(c) illustrates the case of $\beta_1<\zeta(1,2)<\alpha_2$.

When τ is not a Markov transformation there is no simple approach to compute the invariant distribution. In [23], an approximation approach was proposed by using the eigenvector of a certain matrix. A different method is presented in [24] which shows that an expanding piecewise linear map τ can be approximated by a sequence of piecewise linear Markov transformations whose invariant densities converge to the invariant density of τ. Construction details of the piecewise linear Markov approximation sequence can be found in [24].

V. NUMERICAL EXAMPLES

In this section, several examples are given to illustrate the computation of the invariant density when the map τ defined in (2.7) is a Markov transformation. Assume that the initial state is distributed in the interval $[-1,1]$. The optimal 1-bit control law is obtained by setting $d=a-1$. When $a<2$, the trajectory $\{\tau^n([-1,1])\}_{n\geq 0}$ converges to the interval $[-d,d]$. Consider the equation $x^6-x^5-x^4+x^3-x^2-x+1=0$, which has a real root c such that $c\approx 1.556030$. Given the initial partition $\mathscr{P}=\{[-d,0),[0,d]\}$, $\mathscr{P}$ is a Markov partition of order $r=6$. The induced matrix M_τ associated with the partition is

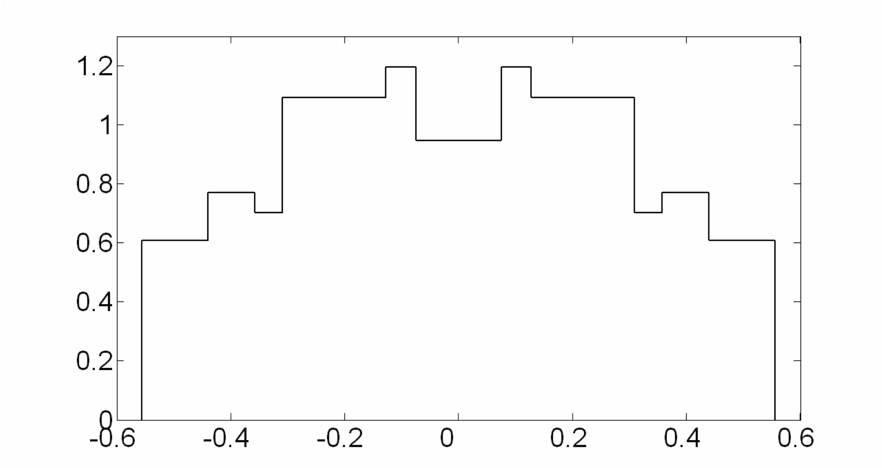

Fig. 3. The invariant density function for c ($c \approx 1.556030$).

$$M_\tau^{\mathrm{T}} = \frac{1}{\mathrm{c}} \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{bmatrix}.$$

The invariant density function is related to the left eigenvector of M_τ and is shown in Fig. 3. Note that c is also a root of the polynomial:

$$a^7 - 2a^6 + 2a^4 - 2a^3 + 2a - 1 = (a^6 - a^5 - a^4 + a^3 - a^2 - a + 1)(a-1) = 0.$$

In view of the notation stated in (4.4), one can show that $\mathbf{v} = [1,2,1,2]$, $\mathbf{s} = [1,3,4,6]$, $m = 4$ and $r = 6$. The sum of the components of $\mathbf{v}$ is equal to 6.

It is shown in Lemma 4.5 that, for any $i \geq 1$, there exists a unique β_i such that $\alpha_i < \beta_i < \alpha_{i+1}$ and there exist infinitely many Markov transformations in the left and the right neighborhoods of β_i. It is hence of interest to study the invariant distribution when the system parameter is close to β_i. In particular, consider the case $\beta_1 = \sqrt{2} = 1.4142\ldots$, which is a root of the equation $a^2 - 1 = 1$. The corresponding Markov partition has 4 elements: $\{[-d, -d/(\beta_1+1)), [-d/(\beta_1+1), 0), [0, d/(\beta_1+1)], (d/(\beta_1+1), d]\}$. The induced

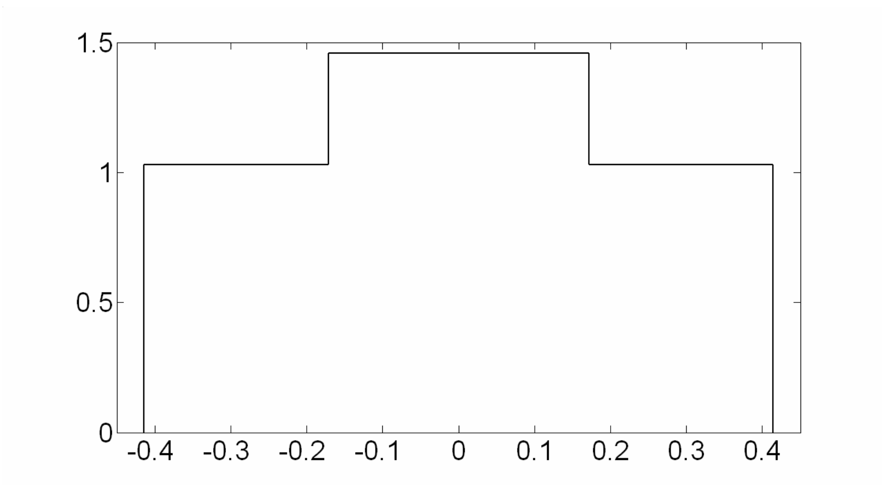

Fig. 4. The invariant density function for $\beta_1 = \sqrt{2}$.

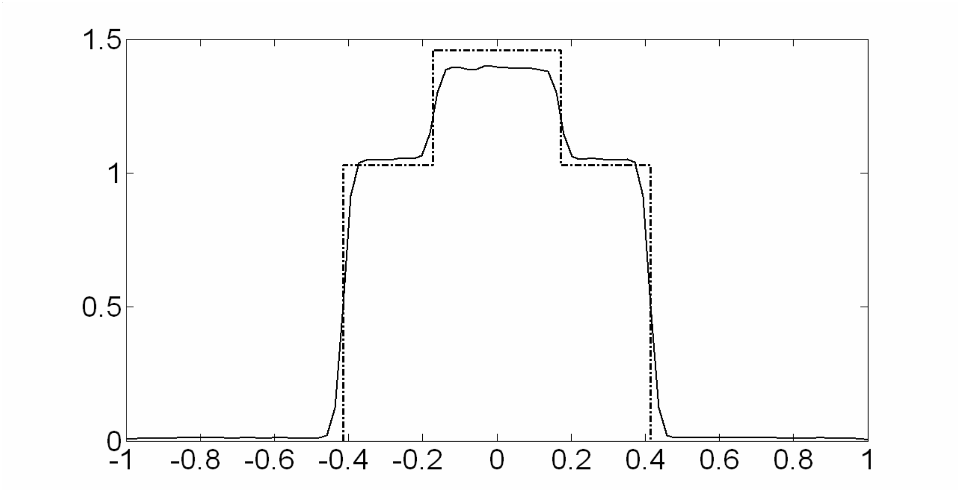

Fig. 5. The invariant density functions for $a = 1.4142$ (solid line) and $\beta_1 = \sqrt{2}$.

matrix M_τ associated with the partition is

$$\mathrm{M}_\tau = \frac{1}{\beta_1}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Fig. 4 shows the invariant density function for this case. The Figures 3 and 4 are obtained from exact computation via Markov transformations. As a comparison, we investigate the invariant density function via computer simulations for a case close to $\beta_1 = \sqrt{2}$. The invariant density function for $a = 1.4142$ is depicted with solid line in Fig. 5, where the invariant distribution for β_1 obtained via exact computation is plotted by dash-dot line. The invariant density function for $a = 1.4142$ was obtained as the histogram of a trajectory via computer simulations. The initial state is chosen from a uniform distribution over the interval $[-1,1]$, and the trajectory of the initial state under the map τ was generated for $1000,000$ iterations. The simulation results appear to be close to those obtained by numerical computation.

VI. Conclusion

This paper analyzes the probabilistic dynamical behavior of control systems under finite communication bandwidth feedback. Based on single bit observations, optimal control laws that minimize asymptotic expected cost function or time-averaged expected cost function are derived. Computation of the invariant distribution is investigated when the map under consideration is a Markov transformation. It is shown that there exist infinite families of solutions such that the resulting closed-loop systems are Markov transformations. As an extension of the problems defined here, it would be of interest to consider models with state and observation noises.

References

[1] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 916-924, 1990.

[2] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth–part I: state estimation problems," *IEEE Trans. on Automat. Contr.*, 42(9), Sept. 1997.

[3] W. S. Wong and R. W. Brockett, "System with finite communication bandwidth constraints-II: stabilization with limited information feedback," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1049-1053, 1999.

[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1279-1289, 2000.

[5] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, 2001.

[6] J. Baillieul, "Feedback designs in information-based control," *Stochastic Theory and Control*, (*Proc. of the workshop held at the University of Kanas*), Springer-Verlag Berlin Heidelberg, Oct. 2002, pp. 35-57.

[7] J. Baillieul, "Feedback coding for information-based control: operating near the data rate limit," *in Proc. 41st Conf. Decision and Control*, Atlanta, USA, Dec. 2002, pp. 3229-3236.

[8] E. Verriest and M. Egerstedt, "Control with delayed and limited information: a first look," *in Proc. 41st Conf. Decision and Control*, Atlanta, USA, Dec. 2002, pp. 1231-1236.

[9] F. Fagnani and S. Zampieri, "Stability analysis and synthesis for scalar linear systems with a quantized feedback," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 1569-1584, 2003.

[10] G. N. Nair and R. J. Evans, "Exponential stabilization of finite-dimensional linear system with limited data rates," *Automatica*, vol. 39, pp. 585-593, 2003.

[11] S. Tatikonda and S. K. Mitter, "Control under communication constraints," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1056-1068, 2004.

[12] F. Fagnani and S. Zampieri, "Quantized stabilization of linear systems: complex versus performance," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1534-1548, 2004.

[13] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (DFCB) control," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1573-1584, 2004.

[14] J. C. Delvenne, "An optimal quantized feedback strategy for scalar linear systems," *IEEE Trans. Automat. Contr.*, vol. 51, pp. 298-303, 2006.

[15] G. N. Nair and R. J. Evans, "Optimal control under a data rate constraint," Proc. 14th Sym. Sys. Ident., Newcastle, Australia, 2006.

[16] N. C. Martins, M. A. Dahleh and N. Elia, “Feedback stabilization of uncertain systems in the presence of a direct link,” *IEEE Trans. Automat. Contr.*, vol. 51, pp. 438-447, 2006.
[17] A. Sahai and S. Mitter, “The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link-part I: scalar system,” *IEEE Trans. Inf. Theory*, vol. 52, pp. 3369-3395, 2006.
[18] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, “Feedback control under data rate constraints: an overview,” *Proc. of IEEE*, vol. 95, no. 1, pp. 108-137, 2007.
[19] Fu M. and Xie L., “Finite-level quantized feedback control for linear systems”, *IEEE Trans. Automat. Contr.*, vol. 54, pp. 1165-1170, 2009.
[20] R. W. Brockett, “Minimum attention control,” *in Proc. 36th Conf. Decision and Control*, San Diego, USA, 1997, pp. 2628-2632.
[21] A. Lasota and J. A. Yorke, “On the existence of invariant measures for piecewise monotonic transformations,” *Trans. Amer. Math. Soc.*, vol. 186, pp. 481-488, 1973.
[22] T. Y. Li and J. A. Yorke, “Ergodic transformations from an interval into itself,” *Trans. Amer. Math. Soc.,* vol. 235, pp. 183-192, 1978.
[23] T. Y. Li, “Finite approximation for the Frobenius-Perron operator. A solution to Ulam’s conjecture,” *Jour. App. Theory,* vol. 17, pp. 177-186, 1976.
[24] P. Gora and A. Boyarsky, “Compactness of invariant densities for families of expanding piecewise monotonic transformations,” *Can. Jour. Math.*, pp. 855-869, 1989.
[25] A. Boyarsky and P. Gora, *Laws of chaos: invariant measures and dynamical systems in one dimension*, Bierkhauser Boston, 1997.
[26] A. Lasota and M. C. Mackey, *Chaos, fractals, and noise: stochastic aspects of dynamics*, 2nd edition, Springer-Verlag, 1994.
[27] R. C. Robinson, *An introduction to dynamical systems: continuous and discrete*, Pearson Prentice Hall, 2004.
[28] T. Stojanovski and L. Kocarev, “Chaos-based random number generators-part I: analysis,” IEEE Trans. Circuits *Syst. I*, vol. 48, pp. 281-288, 2001.

APPENDIX

A. Proof of Lemma 4.1

Proof is by induction. The case for $i=2$ can be verified by direct checking. Assume that the statement is true for *i* up to $j-1$. It follows that

$$f_j(\alpha_{j-1}) = \alpha_{j-1}^{j+1} - 2\alpha_{j-1}^{j} + 1 = (\alpha_{j-1}^{j} - 2\alpha_{j-1}^{j-1} + 1)\alpha_{j-1} - \alpha_{j-1} + 1 < 0. \quad \text{(A1)}$$

On the other hand $f_j(x) \geq 1$ for $x \geq 2$. So the largest real root of $f_j(x)$, α_j, lies in the interval $\alpha_{j-1} < x < 2$. Moreover,

$$0 = \alpha_j^{j+1} - 2\alpha_j^{j} + 1 \Rightarrow \alpha_j^{j} = (\alpha_j^{j-1} - 1)/(\alpha_j - 1) = \alpha_j^{j-1} + \cdots + 1 > j. \quad \text{(A2)}$$

From this, one can show

$$f_j'(x) = (j+1)x^{j} - 2jx^{j-1} = j(x^{j} - 2x^{j-1} + 1) + x^{j} - j > 0, \quad \text{(A3)}$$

for $x \geq \alpha_j$. Hence, $f_j(x) > 0$ for $x > \alpha_j$. □

B. Proof of Lemma 4.2

The fact that $g_i(x) > 0$ for $x > \alpha_i > 1$ follows from Lemma 4.1. Note that this implies,

$$x^i - (x^{i-1} + \cdots + x) > 1. \tag{B1}$$

On the other hand, one can show that for $x \geq \alpha_i$

$$\begin{aligned} &g_i'(x) = ix^{i-1} - [(i-1)x^{i-2} + \cdots + 1] \\ &> i[x^{i-1} - (x^{i-2} + \cdots + 1)] = \frac{i}{x}[x^i - (x^{i-1} + \cdots + x)] > \frac{i}{x}. \end{aligned} \tag{B2}$$

□

C. Proof of Lemma 4.4

Part (i) is straightforward. For (ii) note that since $0 < x < \dfrac{d}{a+1}$, $\tau_a(x) = ax - d < -\dfrac{d}{a+1} < 0$. Hence,

$$\tau_a^2(x) = a^2 x - ad + d \geq \frac{a^2 d}{a+1} - d(a-1) = \frac{d}{a+1} > 0.$$

$$\tau_a^2(x) - x = a^2 x - ad + d - x = (a-1)[(a+1)x - d] < 0.$$

(iii) can be verified by direct checking. □

D. Proof of Lemma 4.5

Proof: Define $h(x) = g_i(x) - 1/(x+1)$. This is a continuous function for $x \geq \alpha_i$. Moreover, $h(\alpha_i) = -1/(\alpha_i + 1) < 0$ and

$$\begin{aligned} &h(\alpha_{i+1}) = g(\alpha_{i+1}) - 1/(\alpha_{i+1} + 1) \\ &= \tau^i_{\alpha_{i+1}}(d)/d - 1/(\alpha_{i+1} + 1) = 1/\alpha_{i+1} - 1/(\alpha_{i+1} + 1) > 0. \end{aligned} \tag{D1}$$

The last equality follows from the fact that $0 = \tau^{i+1}_{\alpha_{i+1}}(d) = \alpha_{i+1}\tau^i_{\alpha_{i+1}}(d) - d$. By the Mean Value Theorem there exists a β_i, $\alpha_i < \beta_i < \alpha_{i+1}$, satisfying (4.11). It follows from Lemma 4.2 that for $x \geq \alpha_i$

$$h'(x) = g_i'(x) + \frac{1}{(x-1)^2} > 0. \tag{D2}$$

Hence, β_i is uniquely defined. □

E. Proof of Proposition 4.2

Proof is by induction. First, for $i \geq 2$ one shows the existence of $\omega(i,1)$ satisfying the statement of Proposition 4.2.

To prove this, recall that $\tau_{\beta_i}^{i}(d) = dg_i(\beta_i) = d/(\beta_i + 1)$. For $x \in [\alpha_i, \beta_i]$, $\tau_x^{i+2}(d)$ is a continuous function in x of the form

$$\tau_x^{i+2}(d) = \tau_x^2(\tau_x^i(d)) = x^2 \tau_x^i(d) - (x-1)d \, . \tag{C1}$$

By (C1) or the comments after Lemma 4.5, one can show that $\tau_{\beta_i}^{i+2}(d) = d/(\beta_i + 1) > 0$. (C1) also implies $\tau_{\alpha_i}^{i+2}(d) = -(\alpha_i - 1)d < 0$ since $i \ge 2$. The existence of $\omega(i,1)$ satisfying $\tau_{\omega(i,1)}^{i+2}(d) = 0$ is then guaranteed by the Mean Value Theorem. Since $\tau_x^{i+2}(d)$ is a polynomial in x in $[\alpha_i, \beta_i]$, one can select $\omega(i,1)$ so that $\tau_x^{i+2}(d) > 0$ for $x \in (\omega(i,1), \beta_i]$. (For $i = 1$, $\tau_x^{i+2}(d)$>0 in the interval $(\alpha_1, \beta_1]$, explaining why this case is different.)

For any $i \ge 2$, assume now that for any k with $j > k \ge 1$, $\omega(i,k)$ satisfies the statement in Proposition 4.2. One claims that this implies the existence of $\omega(i,j)$ satisfying the statement of Proposition 4.2. Recall that

$$\alpha_i < \omega(i,1) < \cdots < \omega(i,j-1) < \beta_i \, . \tag{C2}$$

On the other hand, for any $x \in [\omega(i,j-1), \beta_i]$

$$0 < \tau_x^i(d) \le d/(x+1) \, . \tag{C3}$$

The second inequality follows from Lemma 4.5, while the first inequality follows from Lemma 4.2 since $\alpha_i < \omega(i,j-1)$ according to (C2). It follows from Lemma 4.4 that

$$\tau_x^{i+2}(d) < \tau_x^i(d) \le d/(x+1) \, . \tag{C4}$$

Since $x > \omega(i,1)$, by induction assumption,

$$0 < \tau_x^{i+2}(d) \, . \tag{C5}$$

One can repeat this argument to show:

$$0 < \tau_x^{i+2j-2}(d) < \cdots < \tau_x^{i+2}(d) < \tau_x^i(d) \le d/(x+1) \, . \tag{C6}$$

By Lemma 4.4(i), it follows that the function $\tau_x^{i+2j}(d)$ is of the form

$$\tau_x^{i+2j}(d) = \tau_x^2(\tau_x^{i+2j-2}(d)) = x^2 \tau_x^{i+2j-2}(d) - xd + d \, . \tag{C7}$$

Hence, $\tau_x^{i+2j}(d)$ is continuous on $[\omega(i,j-1), \beta_i]$. Note that

$$\tau_{\beta_i}^{i+2j}(d) = \tau_{\beta_i}^{2j}(\tau_{\beta_i}^i(d)) = \tau_{\beta_i}^i(d) > 0 \, . \tag{C8}$$

Since $\tau^{i+2j-2}_{\omega(i,j-1)}(d)=0$, (C7) implies

$$\tau^{i+2j}_{\omega(i,j-1)}(d)=-(x-1)d<0. \tag{C9}$$

The existence of $\omega(i,j)$ satisfying $\tau^{i+2j}_{\omega(i,j)}(d)=0$ is guaranteed by the Mean Value Theorem. Since $\tau^{i+2j}_{x}(d)$ is a polynomial in x in the interval $[\omega(i,j-1),\beta_i]$, one can select $\omega(i,j)$ so that $\tau^{i+2j}_{x}(d)>0$ for $x\in(\omega(i,j),\beta_i]$. This completes the proof of the stated claim.

The case for $i=1$ can be proven in a similar way and details are omitted. □