Concavity of the Feasible Signal-to-Noise Ratio Region in Power Control Problems

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Abstract

Signal-to-noise (SNR) ratio is commonly regarded as a reliable measure of the performance of wireless communication systems. Knowledge of the fundamental properties of the feasible SNR region can facilitate the performance optimization of multi-user wireless systems. This paper examines the concavity of the feasible SNR region. In particular, it is shown that for systems with only three users, the feasible SNR region is always concave. As concavity for two-dimensional systems is well-known and concavity for four-dimensional systems does not hold in general, this result fills in a gap on this issue. A concavity result for systems with a general number of users is also established under certain technical conditions.

Keyword: Power control, Feasible SNR region, Concavity set

1. Introduction

Signal-to-noise ratios are key performance indicators for wireless communication systems. The signal-to-noise ratio (SNR) of a user is defined by dividing the power of the received signal by the sum of the power from all noises, including interfering transmissions from other users and thermal noises. These ratios are typically analyzed under the assumption that interfering noises form Gaussian processes. Although there are limitations of this assumption, as reported in Sunay and McLane [1, 2] and Chen and Wong [2], the importance of these ratios as a performance measure for wireless systems has never been doubted.

For a wireless system with N users, the signal-to-noise ratios form an N-dimensional vector. In this paper, all users are assumed to be transmitting with strictly positive power; hence all the ratios are strictly positive and the SNR vector is a point lying in the positive orthant, $\mathbb{R}_+^N = \left\{(x_1,\dots x_N) \in \mathbb{R}^N : x_i > 0, \forall i\right\}$. The feasible region defined by the SNR vectors is the focus of this study. For the case where there are only two users, the feasible region can be easily described as the region in $\mathbb{R}_+^2$ bounded by the two axes and the curve:

$$\{(ax, b/x) : x \in \mathbb{R}_+\} \tag{1.1}$$

for some positive a and b whose values are dependent on the channel gains. This feasible region is obviously concave in the sense that its complement in $\mathbb{R}_+^2$ is convex, so that the line joining any two points in the complement is contained in the complement.

It is natural to ask whether this concavity property of the feasible region holds in higher dimensional systems. This has been a longstanding open question. Sung [3] provided an interesting insight into this problem by showing that the feasible region is log-convex. Subsequent papers dealing with geometric properties of the feasibility region include Catrein et. al. [4], Boche and Stańczak [5, 6], Imhof and Mathar [7, 8], Stańczak and Boche [9]. In particular, it has been shown in [9] that the concavity result does not hold for a general four user wireless system. The situation for a general three user system remains open.

In this paper, we investigate the concavity of feasible SNR regions. Main results include the following:

1. The feasible SNR region of a general three user system is concave.

2. Consider a system with a channel gain matrix that can be represented as a product of a diagonal matrix with a symmetric matrix. If the gain matrix has only one eigenvalue strictly greater than 1 and the remaining eigenvalues strictly less than 1, the feasible SNR region is concave.

We will demonstrate by concrete examples that for systems with gain matrices not satisfying the stated eigenvalue condition the feasible SNR region is not necessarily concave. In fact, the geometry of a general feasible SNR region can be extremely complicated. For example, given any two points, *x* and *y*, on the boundary that divides the feasible region and its complement in $\mathbb{R}_{+}^{N}$, the line:

$$\{\alpha x+\beta y:\alpha+\beta=1,0<\alpha,\beta<1\} \tag{1.2}$$

does not necessarily lie completely outside nor inside the feasible SNR region. We will show by numerical examples that the line can be divided into segments, some of which are outside and the remaining ones are inside of the feasible SNR region.

The main tool used in establishing these results originates from an idea proposed in a paper by Weinberger [10] in which he provided a simple proof of a theorem of Lax [11]. The essence of this idea lies in the construction of a special two-variable characteristic polynomial. However, beyond this basic contraption, there is little relation between the arguments used in [10] and this paper.

The organization of the rest of the paper is as follows. Section 2 presents a description of the basic model and assumptions. In Section 3, the two-variable polynomial alluded to earlier is introduced and some of its fundamental properties are described. Moreover, a basic concavity result for a general number of users is established. In Section 4, concavity for a general three-dimensional system is proven. Numerical examples are provided in Section 5 and Section 6 offers some concluding remarks.

2. Basic Model and Background Information

The basic model discussed in this paper consists of N transmitter-receiver pairs, all sharing the same radio spectrum. Let $G_{ij}\geq 0$ represent the channel gain between the *j*-th transmitter and the *i*-th receiver. The $N\times N$ channel gain matrix, $\mathbf{G}$, is defined to be $\left(G_{ij}\right)$. (For any matrix or vector, $\mathbf{M}$, we use the notation $\mathbf{M}\geq 0$ to denote that all components of $\mathbf{M}$ are nonnegative, $\mathbf{M}>0$ to denote that $\mathbf{M}\geq 0$

and $\mathbf{M} \neq 0$, and $\mathbf{M} \gg 0$ to denote that all the components are $\mathbf{M}$ strictly positive.) In practice, $\mathbf{G} \gg 0$. However, to establish the results in this paper, it is sufficient to assume that the interference matrix, $\mathbf{Z} = (Z_{ij})$, defined by

$$\mathbf{Z} = \begin{pmatrix} G_{11}^{-1} & 0 & \cdots & 0 \\ 0 & G_{22}^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{NN}^{-1} \end{pmatrix} \mathbf{G} - \mathbf{I} \tag{2.1}$$

is primitive [12].

Let $\mathbf{p} = (p_1, ..., p_m) \gg 0$ denote a power vector and $\boldsymbol{\eta} = (\eta_1, ..., \eta_m) \gg 0$ a noise power vector. The signal-to-noise ratio for the *i*-th user, represented by $\Gamma_i(\mathbf{p})$, is defined by

$$\Gamma_i(\mathbf{p}) = \frac{G_{ii} p_i}{\sum_{j \neq i} G_{ij} p_j + \eta_i}. \tag{2.2}$$

Let $\boldsymbol{\Gamma}(\mathbf{p}) = \left[\Gamma_1(\mathbf{p}), \ldots, \Gamma_N(\mathbf{p})\right]^T$ represent an *N*-dimensional column SNR vector. It is easy to check that the SNR vector is a function of $\mathbf{Z}$ and one can define the feasible SNR region in terms of either $\mathbf{G}$ or $\mathbf{Z}$. More specifically, one can view $\boldsymbol{\Gamma}(\mathbf{p})$ as a continuous mapping from $\mathbb{R}_+^N$ to itself. For a given interference matrix $\mathbf{Z}$, the *feasible SNR region*, $\mathcal{F}(\mathbf{Z})$, is defined to be the closure of the image set $\boldsymbol{\Gamma}(\mathbb{R}_+^N)$ in $\mathbb{R}_+^N$. Let $S(\mathbf{Z})$ denote the set of limit points $\lim_{c \to \infty} \boldsymbol{\Gamma}(c\mathbf{p})$ for some $\mathbf{p} \in \mathbb{R}_+^N$. $S(\mathbf{Z})$ contains elements of the form

$$\left[\frac{p_1}{\sum_j Z_{1j} p_j}, \ldots, \frac{p_N}{\sum_j Z_{Nj} p_j}\right] \tag{2.3}$$

for some $(p_1, ..., p_N) \gg 0$.

Based on previous works reported in the literature, the structure of a feasible SNR region, $\mathcal{F}(\mathbf{Z})$, can be understood in the following way [3]. For any point $\mathbf{z}$ in $S(\mathbf{Z})$, the open line segment defined by $\{c\mathbf{z} : 0 < c < 1\}$ is in $\boldsymbol{\Gamma}(\mathbb{R}_+^N)$, but $c\mathbf{z}$ is not in the feasible region if $c > 1$. It follows that $S(\mathbf{Z})$ is in the boundary set of the feasible SNR region.

Definition: An SNR feasible region, $\mathcal{F}(\mathbf{Z})$, is concave if for any two distinct points, $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$, on the boundary $S(\mathbf{Z})$, the open line

$$\{\alpha\boldsymbol{\xi} + \beta\boldsymbol{\psi} : \alpha + \beta = 1, 0 < \alpha, \beta < 1\} \tag{2.4}$$

is outside of $\mathcal{F}(\mathbf{Z})$.

For subsequent discussions, let $\rho(\mathbf{M})$ represent the spectral radius of a matrix $\mathbf{M}$. If $\boldsymbol{\Gamma} = [\Gamma_1, \ldots, \Gamma_N]^T$ is an SNR vector, let $\mathbf{D}_{\boldsymbol{\Gamma}}$ denote the corresponding diagonal matrix

$$\begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \Gamma_N \end{pmatrix}. \tag{2.5}$$

Under this notation, $\boldsymbol{\Gamma}$ is contained in $S(\mathbf{Z})$ if and only if there exists a power vector, $\mathbf{p} \gg 0$ such that

$$\mathbf{D}_{\boldsymbol{\Gamma}}\mathbf{Z}\mathbf{p} = \mathbf{p}. \tag{2.6}$$

3. Basic Analysis

Consider a fixed interference matrix, $\mathbf{Z}$, with a feasible region $\mathcal{F}(\mathbf{Z})$. Let $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ be distinct elements on the boundary $S(\mathbf{Z})$ and let $\mathbf{D}_{\boldsymbol{\xi}}$ and $\mathbf{D}_{\boldsymbol{\psi}}$ represent the corresponding diagonal matrices. From (2.6) it follows that

$$\rho(\mathbf{D}_{\boldsymbol{\xi}}\mathbf{Z}) = \rho(\mathbf{D}_{\boldsymbol{\psi}}\mathbf{Z}) = 1. \tag{3.1}$$

For $\alpha \in (0,1), \beta = 1 - \alpha$ let $\mathbf{D}_{\boldsymbol{\theta}}$ be the diagonal matrix corresponding to the point

$$\boldsymbol{\theta} = \frac{1}{\rho(\alpha\mathbf{D}_{\boldsymbol{\xi}}\mathbf{Z} + \beta\mathbf{D}_{\boldsymbol{\psi}}\mathbf{Z})}(\alpha\boldsymbol{\xi} + \beta\boldsymbol{\psi}). \tag{3.2}$$

It follows from definition that

$$\mathbf{D}_{\boldsymbol{\theta}} = \frac{1}{\rho(\alpha\mathbf{D}_{\boldsymbol{\xi}}\mathbf{Z} + \beta\mathbf{D}_{\boldsymbol{\psi}}\mathbf{Z})}(\alpha\mathbf{D}_{\boldsymbol{\xi}} + \beta\mathbf{D}_{\boldsymbol{\psi}}). \tag{3.3}$$

Since the spectral radius of $\mathbf{D}_{\boldsymbol{\theta}}\mathbf{Z}$ is exactly 1, from the Perron-Frobenius Theorem, (2.6) has a strictly positive solution, and hence $\boldsymbol{\theta}$ is in $S(\mathbf{Z})$. Due to the structure of the feasible region, we conclude that the point $\alpha\boldsymbol{\xi} + \beta\boldsymbol{\psi}$ lies outside of the feasible region if and only if

$$\rho(\alpha\mathbf{D}_{\boldsymbol{\xi}}\mathbf{Z} + \beta\mathbf{D}_{\boldsymbol{\psi}}\mathbf{Z}) > 1 = \alpha\rho(\mathbf{D}_{\boldsymbol{\xi}}\mathbf{Z}) + \beta\rho(\mathbf{D}_{\boldsymbol{\psi}}\mathbf{Z}). \tag{3.4}$$

A mathematical tool of central importance in this paper is the following two-variable polynomial:

$$p(x,y) = \det\left(y(\mathbf{D}_\xi \mathbf{Z} - \mathbf{I}) + \mathbf{D}_\psi \mathbf{Z} - x\mathbf{I}\right) \tag{3.5}$$

which was adopted from [10] and was originally used to prove a theorem of Lax in regard to concavity of the eigenvalue function. The real roots of this polynomial are elements (x,y) in $\mathbb{R}^2$ such that $p(x,y)=0$. Some salient features of these real roots are summarized in the following lemma.

Lemma 3.1: For any x there are at most $N-1$ real roots. For any non-negative y there is at least one real root (x,y) with $x>0$. In particular, the point $(1,0)$ is a root of p and there is no root of the form $(x,0)$ for $x>1$.

Proof: The coefficient of y^N in $p(x,y)$ is $\det(\mathbf{D}_\xi \mathbf{Z} - \mathbf{I})$ which is equal to zero since ξ in $S(\mathbf{Z})$. So the first statement holds. For any non-negative y,

$$\rho\left(y\mathbf{D}_\xi \mathbf{Z} + \mathbf{D}_\psi \mathbf{Z}\right) > \rho\left(y\mathbf{D}_\xi \mathbf{Z}\right) = y. \tag{3.6}$$

The first inequality follows from a well-known property of irreducible matrices, (see for example [12].) There is a real root, (x,y), for p with

$$x = \rho\left(y(\mathbf{D}_\xi \mathbf{Z} - \mathbf{I}) + \mathbf{D}_\psi \mathbf{Z}\right) = \rho\left(y\mathbf{D}_\xi \mathbf{Z} + \mathbf{D}_\psi \mathbf{Z}\right) - y > 0 \tag{3.7}$$

and there is no other root, (x',y), with $x' > x$. ■

For any non-negative y, let

$$m(y) = \rho\left(y\mathbf{D}_\xi \mathbf{Z} + \mathbf{D}_\psi \mathbf{Z}\right) - y. \tag{3.8}$$

The previous proof shows that $(m(y),y)$ is a real root of p and there is no root of p, (x,y), such that

$$x > m(y). \tag{3.9}$$

(See Figure 1a for an illustration.) The usefulness of p in proofing concavity result was realized in [10] and is reflected by the following lemma.

Lemma 3.2: Let $\alpha,\beta>0$, then

$$\begin{aligned} \rho\left(\alpha\mathbf{D}_\xi \mathbf{Z} + \beta\mathbf{D}_\psi \mathbf{Z}\right) &< \alpha\rho\left(\mathbf{D}_\xi \mathbf{Z}\right) + \beta\rho\left(\mathbf{D}_\psi \mathbf{Z}\right) \quad \text{if } m(\alpha/\beta) < 1, \\ \rho\left(\alpha\mathbf{D}_\xi \mathbf{Z} + \beta\mathbf{D}_\psi \mathbf{Z}\right) &> \alpha\rho\left(\mathbf{D}_\xi \mathbf{Z}\right) + \beta\rho\left(\mathbf{D}_\psi \mathbf{Z}\right) \quad \text{if } m(\alpha/\beta) > 1. \end{aligned} \tag{3.10}$$

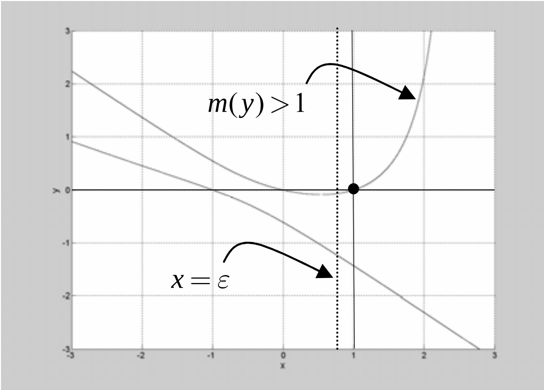

Figure 1a: $m(y)>1$ implies concavity

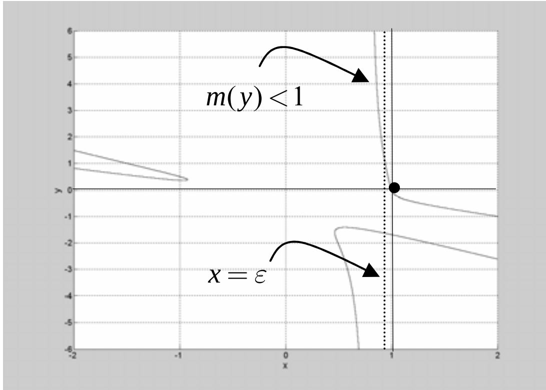

Figure 1b: $m(y)<1$ implies convexity

Proof: Since $\rho\left(\mathbf{D}_{\xi}\mathbf{Z}\right)=\rho\left(\mathbf{D}_{\psi}\mathbf{Z}\right)=1$, it follows from (3.8) that for any $y\geq 0$

$$\begin{aligned}\rho\left(y\mathbf{D}_{\xi}\mathbf{Z}+\mathbf{D}_{\psi}\mathbf{Z}\right)&<y\rho\left(\mathbf{D}_{\xi}\mathbf{Z}\right)+\rho\left(\mathbf{D}_{\psi}\mathbf{Z}\right) \quad \text{if } m(y)<1,\\ \rho\left(y\mathbf{D}_{\xi}\mathbf{Z}+\mathbf{D}_{\psi}\mathbf{Z}\right)&>y\rho\left(\mathbf{D}_{\xi}\mathbf{Z}\right)+\rho\left(\mathbf{D}_{\psi}\mathbf{Z}\right) \quad \text{if } m(y)>1.\end{aligned} \tag{3.11}$$

The lemma follows by letting $y=\alpha/\beta$ and multiplying both sides of the inequality in (3.11) by β. ■

Lemma 3.3: Consider a symmetric $\mathbf{S}$ and a diagonal matrix $\mathbf{E}$ with strictly positive diagonal elements. The matrices $\mathbf{S}$ and $\mathbf{ES}$ have the same number of strictly positive (and strictly negative) eigenvalues.

Proof: Since $\det\mathbf{E}\neq 0$, the matrices $\mathbf{ES}$ and $\mathbf{E}^{1/2}\mathbf{SE}^{1/2}$ have the same characteristics equation and hence they have the same eigenvalues. On the other hand, by the Law of Inertia for quadratic forms [13] the number of strictly positive eigenvalues for $\mathbf{S}$ and $\mathbf{E}^{1/2}\mathbf{SE}^{1/2}$ is identical. Similar result holds for strictly negative eigenvalues. ■

Theorem 3.4: Suppose $\mathbf{Z}$ is of the form $\mathbf{ES}$ where $\mathbf{E}$ is a diagonal matrix and $\mathbf{S}$ is symmetric. If $\mathbf{Z}$ is non-singular and has only one strictly positive eigenvalue, then for any points $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ in the boundary, $S(\mathbf{Z})$, the polynomial in y, $\det(y(\mathbf{D}_{\xi}\mathbf{Z}-\mathbf{I})+\mathbf{D}_{\psi}\mathbf{Z}-\mathbf{I})$, has only non-positive real roots. Moreover, the feasible SNR region of $\mathbf{Z}$ is concave.

Proof: Let $\mathbf{Y}=\mathbf{D}_{\psi}^{1/2}\mathbf{E}^{1/2}\mathbf{SE}^{1/2}\mathbf{D}_{\psi}^{1/2}$. By the Law of Inertia for quadratic forms, the matrices $\mathbf{S}$, $\mathbf{E}^{1/2}\mathbf{S}\mathbf{E}^{1/2}$, and $\mathbf{Y}$ have the same number of strictly positive (and strictly negative) eigenvalues. Since $\mathbf{E}^{1/2}\mathbf{S}\mathbf{E}^{1/2}$ and $\mathbf{ES}=\mathbf{Z}$ have the same eigenvalues, it follows that $\mathbf{Y}$ has one strictly positive eigenvalue and $N-1$ strictly negative eigenvalues. Let $\mathbf{D}=\mathbf{D}_\xi\mathbf{D}_\psi^{-1}$ then

$$\begin{aligned} q_x(y) &= \det(y(\mathbf{D}_\xi\mathbf{Z}-\mathbf{I})+\mathbf{D}_\psi\mathbf{Z}-x\mathbf{I}) \\ &= \det(y(\mathbf{D}\mathbf{D}_\psi^{1/2}\mathbf{E}^{1/2}\mathbf{S}\mathbf{E}^{1/2}\mathbf{D}_\psi^{1/2}-\mathbf{I})+\mathbf{D}_\psi^{1/2}\mathbf{E}^{1/2}\mathbf{S}\mathbf{E}^{1/2}\mathbf{D}_\psi^{1/2}-x\mathbf{I}) \\ &= \det(y(\mathbf{DY}-\mathbf{I})+\mathbf{Y}-x\mathbf{I}) = \det(y(\mathbf{D}-\mathbf{Y}^{-1})+\mathbf{I}-x\mathbf{Y}^{-1})\det\mathbf{Y}. \end{aligned} \tag{3.12}$$

On the other hand, since $\mathbf{Y}=\mathbf{D}_\psi^{1/2}\mathbf{E}^{1/2}\mathbf{S}\mathbf{E}^{1/2}\mathbf{D}_\psi^{1/2}$ and $\mathbf{D}_\psi\mathbf{ES}=\mathbf{D}_\psi\mathbf{Z}$ have the same set of eigenvalues, 1 is an eigenvalue of $\mathbf{Y}$. If $\varepsilon<1$, $\mathbf{I}-\varepsilon\mathbf{Y}^{-1}$ is a positive definite matrix. By the well-known result on positive definite pencil of symmetric matrices, there exists a real congruent transformation that puts $\mathbf{I}-\varepsilon\mathbf{Y}^{-1}$ and $\mathbf{D}-\mathbf{Y}^{-1}$ into diagonal form simultaneously. In particular, the diagonal form of $\mathbf{I}-\varepsilon\mathbf{Y}^{-1}$ can be assumed to the identity matrix. By the Law of Inertia, the number of strictly negative roots of $q_\varepsilon(y)$ is equal to the number of strictly positive eigenvalues of $\mathbf{D}-\mathbf{Y}^{-1}$, and in turn by Lemma 3.3, is equal to the number of strictly positive eigenvalues of $\mathbf{I}-\mathbf{D}^{-1}\mathbf{Y}^{-1}$, or equivalently $\mathbf{I}-\mathbf{D}^{-1/2}\mathbf{Y}^{-1}\mathbf{D}^{-1/2}$. Note that:

$$\mathbf{D}^{-1/2}\mathbf{Y}^{-1}\mathbf{D}^{-1/2}=\mathbf{D}_\xi^{-1/2}\mathbf{Y}^{-1}\mathbf{D}_\xi^{-1/2}. \tag{3.13}$$

So $\mathbf{D}^{-1/2}\mathbf{Y}^{-1}\mathbf{D}^{-1/2}$ just like $\mathbf{Y}$ has one strictly positive eigenvalue and $N-1$ strictly negative eigenvalues. Moreover, $\mathbf{D}_\xi^{-1/2}\mathbf{Y}^{-1}\mathbf{D}_\xi^{-1/2}$ and $\mathbf{Y}^{-1}\mathbf{D}_\xi^{-1}$ have the same set of eigenvalues which include the value $1=\rho(\mathbf{Z}^{-1}\mathbf{D}_\xi^{-1})=1/\rho(\mathbf{D}_\xi\mathbf{Z})$. Thus, $\mathbf{I}-\mathbf{D}^{-1/2}\mathbf{Y}^{-1}\mathbf{D}^{-1/2}$ has $N-1$ strictly positive eigenvalue and a single eigenvalue at 0. It follows that $q_\varepsilon(y)$ has $N-1$ strictly negative roots and a single root at infinity. Denote the finite roots as

$$r_1(\varepsilon)\le\ldots\le r_{N-1}(\varepsilon)<0 \tag{3.14}$$

As ε tends to 1, by continuity of roots to the characteristic equation, we conclude that there are no strictly positive roots for $q_1(y)$.

Now consider the function $m(y)$ defined in (3.8). Clearly, $m(0)=1$, which corresponds to the fact that $(1,0)$ is a real root for $q_1(y)$. For small enough positive y, $m(y)$ cannot be less than 1, due to the fact that $q_\varepsilon(y)$ cannot have a strictly positive root if ε is slightly less than 1. (See Figures 1b for an illustration.) Thus, $m(y)>1$ holds for all positive y in a small enough neighborhood around 0. As y

increases, the continuous curve $(m(y), y)$ cannot cross the line $x = 1$ as $q_1(y)$ has no strictly positive roots. By Lemma 3.2 and using the condition defined in (3.4), it follows that the feasible region is concave. ■

In Section 5 we will show by numerical examples that if $\mathbf{Z}$ has more than one positive eigenvalue, the feasible SNR region is not concave even if $\mathbf{Z}$ is symmetric.

Since $\mathbf{D}_\xi\mathbf{Z}$ is primitive, it has a simple root at 1, so there exists a non-singular matrix, $\mathbf{Q}$, such that

$$\mathbf{D}_\xi\mathbf{Z} = \mathbf{Q}\begin{pmatrix} 1 & \mathbf{0}_{N-1}^T \\ \mathbf{0}_{N-1} & \mathbf{\Lambda}_0 \end{pmatrix}\mathbf{Q}^{-1}, \tag{3.15}$$

where $\mathbf{\Lambda}_0$ is an $(N-1)\times(N-1)$ sub-matrix and $\mathbf{0}_{N-1}$ and $\mathbf{0}_{N-1}^T$ are $(N-1)\times 1$ and $1\times(N-1)$ zero matrices respectively. It follows from Perron-Frobenious Theorem that $\det(\mathbf{\Lambda}_0 - \mathbf{I}_{N-1}) \neq 0$ and $\operatorname{sgn}\det(\mathbf{\Lambda}_0 - \mathbf{I}_{N-1}) = (-1)^{N-1}$. If we write

$$\mathbf{Q} = [\mathbf{p}, \mathbf{R}_0], \quad \mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{q}^T \\ \mathbf{S}_0 \end{bmatrix}, \tag{3.16}$$

Then it follows directly from (3.15) that:

$$\mathbf{D}_\xi\mathbf{Z}\mathbf{p} = \mathbf{p}, \quad \mathbf{q}^T\mathbf{D}_\xi\mathbf{Z} = \mathbf{q}^T, \quad \mathbf{q}^T\mathbf{p} = 1. \tag{3.17}$$

Lemma 3.5: Under the previous notation,

$$\begin{aligned} p(x,y) &= \det\left(y(\mathbf{D}_\xi\mathbf{Z} - \mathbf{I}) + \mathbf{D}_\psi\mathbf{Z} - x\mathbf{I}\right) \\ &= \det\begin{bmatrix} \mathbf{q}^T\mathbf{D}\mathbf{p} - x & \mathbf{q}^T\mathbf{D}\mathbf{Q}_0\mathbf{\Lambda}_0 \\ \mathbf{S}_0\mathbf{D}\mathbf{p} & y(\mathbf{\Lambda}_0 - \mathbf{I}_{N-1}) - x\mathbf{I}_{N-1} + \mathbf{S}_0\mathbf{D}\mathbf{Q}_0\mathbf{\Lambda}_0 \end{bmatrix}. \end{aligned} \tag{3.18}$$

This can also be verified by direct computation and details are omitted here.

Proposition 3.6: Consider a general interference matrix, $\mathbf{Z}$. If the polynomial in y

$$q_1(y) = \det(y(\mathbf{D}_\xi\mathbf{Z} - \mathbf{I}) + \mathbf{D}_\psi\mathbf{Z} - \mathbf{I}) = c_{n-1}y^{N-1} + \ldots + c_1 y \tag{3.19}$$

has only non-positive roots, then for all non-negative y, the property $m(y) > 1$ holds if and only if

$$c_{N-1}(-1)^{N-1} > 0. \tag{3.20}$$

Moreover, (3.20) holds if and only if

$$c_1(-1)^{N-1} > 0. \tag{3.21}$$

Proof: Let ε be an arbitrarily small but fixed positive number. Let

$$x_- = \mathbf{q}^T\mathbf{D}\mathbf{p} - \varepsilon, \quad x_+ = \mathbf{q}^T\mathbf{D}\mathbf{p} + \varepsilon. \tag{3.22}$$

If x is equal to x_- or x_+, then for all large enough y the dominant term in the determinant in (3.18) is

$$(\mathbf{q}^T\mathbf{D}\mathbf{p}-x)\det(\mathbf{\Lambda}_0-\mathbf{I}_{N-1})y^{N-1}. \tag{3.23}$$

Under this condition, the sign of $p(x,y)$ is equal to the sign of the expression in (3.23). It follows that $p(x,y)$ changes sign as x varies from x_- to x_+ for all large enough y, hence, $p(x,y)$ has a real root between x_- and x_+. By continuity of the roots, as ε tends to zero, there is root branch that tends to $(\mathbf{q}^T\mathbf{D}\mathbf{p},\infty)$. On the other hand, since

$$p(x,y)=y\det\left((\mathbf{D}_\xi\mathbf{Z}-\mathbf{I})+\frac{1}{y}\mathbf{D}_\psi\mathbf{Z}-\frac{x}{y}\mathbf{I}\right), \tag{3.24}$$

for a fixed, large enough, positive y, there is a one-to-one relation between x roots of $p(x,y)$ with strictly negative real parts and eigenvalues of $\mathbf{D}_\xi\mathbf{Z}-\mathbf{I}$ with strictly negative real parts. Since there are $N-1$ such roots according to Perron-Frobenius Theorem, one can conclude that for large enough, positive y, $p(x,y)$ has one and only one strictly positive real root in x and the root is in a neighborhood of $x=\mathbf{q}^T\mathbf{D}\mathbf{p}$. Moreover, the neighborhood can be made arbitrarily small as y approaches infinity. Thus for large enough positive y, $m(y)>1$ if and only if $\mathbf{q}^T\mathbf{D}\mathbf{p}>1$. By proposition assumption, $m(y)>1$ holds all positive y if and only if it holds for some large enough positive y.

On the other hand, from (3.18) and (3.19) it follows that

$$c_{N-1}=(\mathbf{q}^T\mathbf{D}\mathbf{p}-1)\det(\mathbf{\Lambda}_0-\mathbf{I}_{N-1}). \tag{3.25}$$

Therefore for positive y, $m(y)>1$ if and only if $c_{N-1}(-1)^{N-1}>0$. To prove the last statement, note that

$$\begin{aligned} q_1(y)&=y^Nq_1(1/y)=y^N\det\left(\mathbf{D}_\xi\mathbf{Z}-\mathbf{I}+\frac{1}{y}(\mathbf{D}_\psi\mathbf{Z}-\mathbf{I})\right)\\ &=y^N\left(c_{n-1}/y+\ldots+c_1/y^{N-1}\right). \end{aligned} \tag{3.26}$$

Hence c_1 is the coefficient of the leading term of the following polynomial in z,

$$\det\left(\mathbf{D}_\xi\mathbf{Z}-\mathbf{I}+z(\mathbf{D}_\psi\mathbf{Z}-\mathbf{I})\right). \tag{3.27}$$

One can repeat previous arguments by reversing the role of $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$. In particular, one can define

$$n(z)=\rho\left(\mathbf{D}_\xi\mathbf{Z}+z\mathbf{D}_\psi\mathbf{Z}\right)-z \tag{3.28}$$

as in (3.8). It follows that $m(y)>1$ for all positive y if and only if $n(z)>1$ for positive z. The last statement of the proposition follows by applying previous arguments using $n(z)$ instead of $m(y)$. ■

4. A Concavity Theorem for Three-Dimensional Systems

When $N=2$, the feasible SNR region is concave, that is, the infeasible region in $\mathbb{R}^2_+$ is convex [5, 6]. For $N \geq 4$, the concavity result has been shown in [9] to be invalid in general. In this paper, concavity of the feasible SNR region is established for all three-dimensional systems. Consider first of all the case that $\mathbf{Z}$ is symmetric. Since $\mathbf{Z}$ has all diagonal elements equal to 0 this implies the sum of its eigenvalues is 0. Other than the dominant eigenvalue, the other two eigenvalues must be strictly negative, since each one must be strictly less than the dominant eigenvalue. Theorem 3.4 then implies that for all three-dimensional systems with symmetric normalized interference matrix the feasible SNR region is concave.

Theorem 4.1: The feasible SNR region of a three-dimensional system is concave.

Before proving this result, we need to establish the following lemma for three-dimensional systems.

Lemma 4.2: For any three-dimensional interference matrix, $\mathbf{Z}$, the polynomial in y

$$q(y) = \det(y(\mathbf{D}_\xi \mathbf{Z} - \mathbf{I}) + \mathbf{D}_\psi \mathbf{Z} - \mathbf{I}) \tag{4.1}$$

has a strictly negative root in addition to the root at 0.

Proof: Let

$$\mathbf{D}_\xi \mathbf{Z} = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}, \quad \mathbf{D}_\psi = \mathbf{D}\mathbf{D}_\xi, \quad \mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}. \tag{4.2}$$

From previous remarks, it is clear that $\det \mathbf{D}_\xi \mathbf{Z} > 0$.

Note that the d_i's are strictly positive. If one of the d_i's is equal to 1, then one can show by direct verification that $y=-1$ is a root to $q(y)$, proving the lemma. On the other hand, results from Perron-Frobenius theory implies it is impossible that all d_i's are less than 1. Thus, without lost of generality, we can assume that (after reversing the role of $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ if necessary)

$$d_1 \leq d_2 < 1 < d_3. \tag{4.3}$$

Then,

$$q(-1) = \det(\mathbf{D}_\psi \mathbf{Z} - \mathbf{D}_\xi \mathbf{Z}) = (d_1 - 1)(d_2 - 1)(d_3 - 1)\det \mathbf{D}_\xi \mathbf{Z} > 0. \tag{4.4}$$

On the other hand,

$$q(-d_2) = (d_2 - 1)^3 - (d_2 - 1)(d_1 - d_2)(d_3 - d_2)be < 0. \tag{4.5}$$

So $q(y)$ has a root in the interval $[-1,-d_2]$. ■

Proof of Theorem 4.1: Construct a continuous one-parameter family of matrices by,

$$\mathbf{Z}_r = \mathbf{Z} + r\mathbf{Z}^T . \tag{4.6}$$

Define a family of characteristic polynomials:

$$q_r(y) = \det(y(\mathbf{D}_\xi \mathbf{Z}_r / \rho(\mathbf{D}_\xi \mathbf{Z}_r) - \mathbf{I}) + \mathbf{D}_\psi \mathbf{Z}_r / \rho(\mathbf{D}_\psi \mathbf{Z}_r) - \mathbf{I}) = a_r y^2 + b_r y . \tag{4.7}$$

The coefficient functions, a_r and b_r, are continuous. From the proof of Lemma 4.2, one can see that these functions cannot vanish since any $q_r(y)$ has a root at zero and a strictly negative root. Thus,

$$\mathrm{sgn}(a_0) = \mathrm{sgn}(a_1) > 0 \tag{4.8}$$

The last inequality followings from Proposition 3.6 and the fact that the feasible region of $\mathbf{Z} + \mathbf{Z}^T$ is concave. So the feasible region for $\mathbf{Z}$ is concave also. ■

5. Numerical Examples

In this section, we present some numerical examples to illustrate the results reported previously. First of all, we note a four-dimensional example which satisfies the eigenvalue condition stated in Theorem 3.4. Consider the following system:

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0 & 0.5 \\ 0.1 & 0.1 & 0.5 & 0 \end{bmatrix}. \tag{5.1}$$

The eigenvalues of $\hat{\mathbf{Z}}$ are $(0.6235, -0.0126, -0.1000, -0.5110)$. If

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.2}$$

and $\mathbf{D}_1 = \mathbf{E}_1 / \rho(\mathbf{E}_1 \hat{\mathbf{Z}})$ and $\mathbf{D}_2 = \mathbf{E}_2 / \rho(\mathbf{E}_2 \hat{\mathbf{Z}})$, then Figure 2a show the roots of the polynomial

$$p_1(x, y) = \det(y(\mathbf{D}_2 \hat{\mathbf{Z}} - \mathbf{I}_4) + \mathbf{D}_1 \hat{\mathbf{Z}} - x\mathbf{I}_4) . \tag{5.3}$$

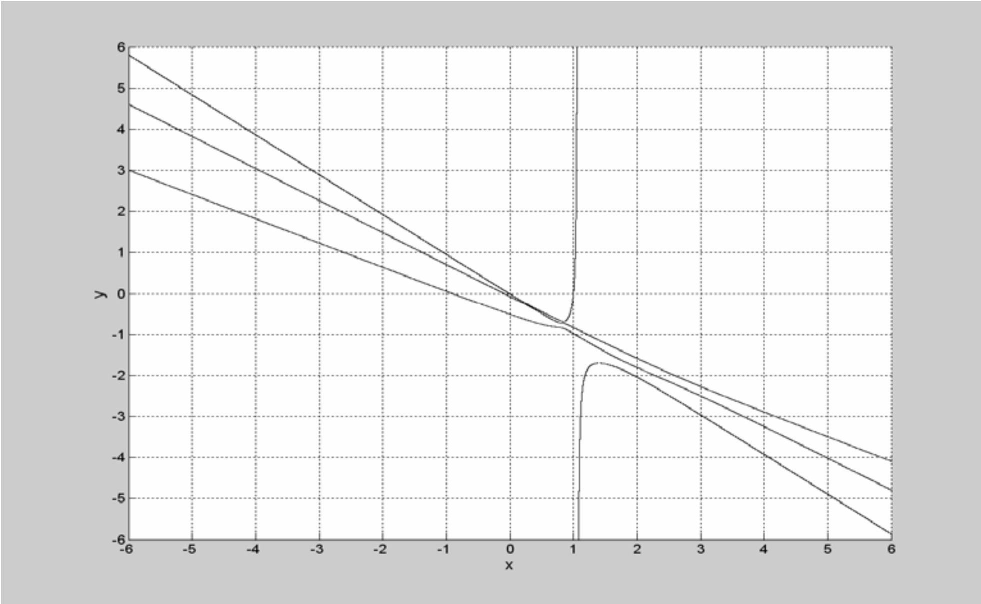

Figure 2a: A four-dimensional symmetric example showing concavity property

If the we redefine $\mathbf{E}_1$ and $\mathbf{E}_2$ so that

$$\mathbf{E}_1 = \begin{bmatrix} .1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \tag{5.4}$$

then the real roots of p_1 are shown in Figure 2b.

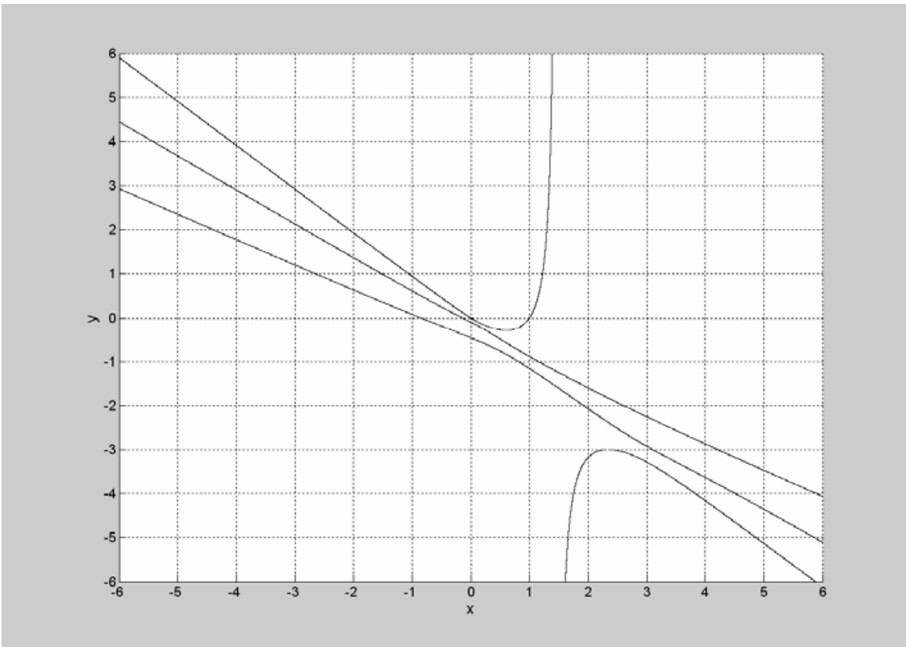

Figure 2b: A four-dimensional symmetric example showing concavity property

We also consider one case where $\mathbf{Z}$ is a non-symmetric matrix with only one eigenvalue with strictly positive real part. Let

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 & 0.11 & 0.3 & 0.3 \\ 0.2 & 0 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.1 & 0.2 & 0.3 & 0 \end{bmatrix}. \tag{5.1}$$

The eigenvalues are $(0.7069, -0.1345+0.1086i, -0.1345-0.1086i, -0.4379)$. For $\mathbf{E}_1$ and $\mathbf{E}_2$ defined as in (5.4), the real roots of p_1 are shown in Figure 3.

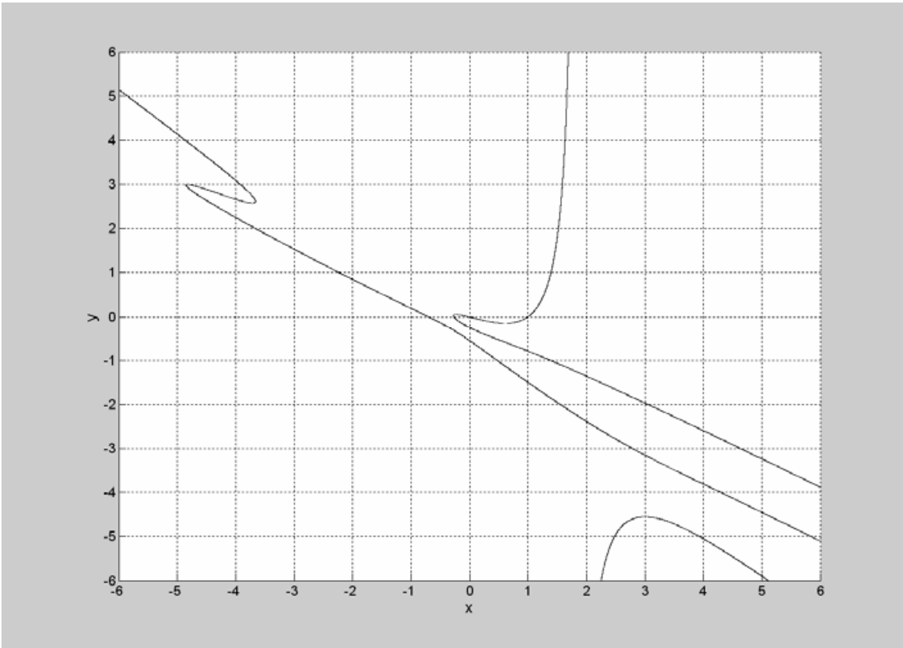

Figure 3: A four-dimensional non-symmetric example showing concavity property

The assumption of a single strictly positive eigenvalue is a key assumption for concavity. Consider the following symmetric case with

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 & 0.2 & 0.01 & 0.02 \\ 0.2 & 0 & 0.2 & 0.01 \\ 0.01 & 0.2 & 0 & 0.3 \\ 0.02 & 0.01 & 0.3 & 0 \end{bmatrix}. \tag{5.2}$$

The eigenvalues are $(0.3952, 0.1363, -0.1534, -0.3781)$. Define $\mathbf{E}_1$ and $\mathbf{E}_2$ as in (5.2), then the real roots of p_1 are shown in Figure 4a.

When one sets

$$\mathbf{E}_1 = \begin{bmatrix} .1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix}, \tag{5.3}$$

then the line segment joining the corresponding points on the boundary of the feasible region exhibits a concavity property as shown in Figure 4b

In fact, it is noted that the concavity property can be made very drastic. If we denote the feasible points as $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$, with

$$\begin{aligned} \boldsymbol{\xi} &= [0.227, 22.729, 0.227, 22.729]^T, \\ \boldsymbol{\psi} &= [23.236, 0.232, 23.236, 0.232]^T, \end{aligned} \tag{5.4}$$

then the feasible point in the direction defined by the midpoint of the line joining $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ is:

$$[1.011, 1.011, 1.011, 1.011]^T. \tag{5.5}$$

The SNR of some of the users are reduced by a factor of around 23.

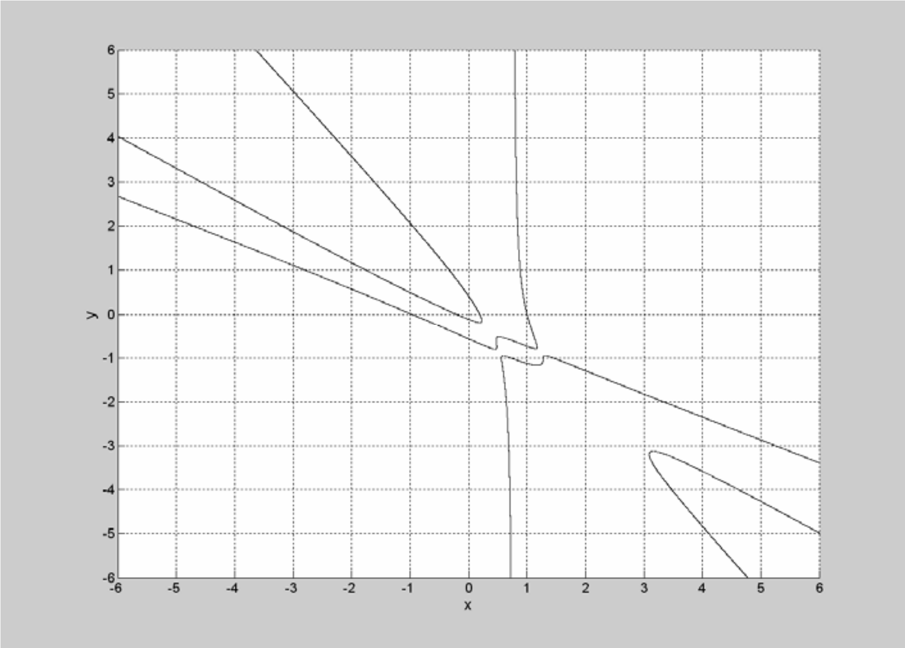

Figure 4a: A four-dimensional symmetric example showing concavity does not hold

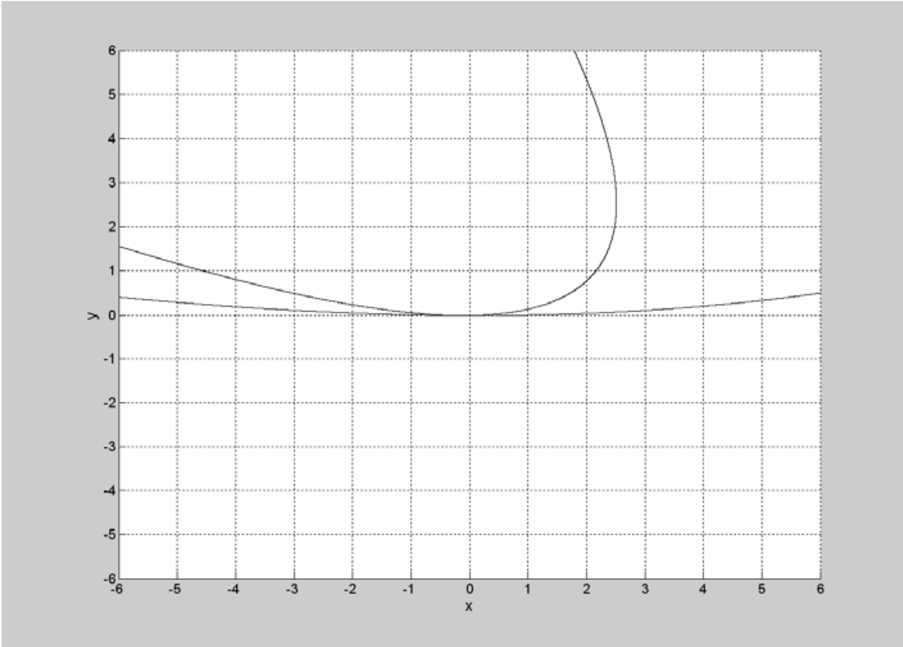

Figure 4b: Same example showing concavity property

Finally, we note with one example that the geometry of the feasible SNR region can be very complex. Consider a five-dimensional system with

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 & 0.8 & 0.4 & 0.008 & 0.08 \\ 0.08 & 0 & 0.8 & 0.004 & 0.008 \\ 0.004 & 0.08 & 0 & 0.08 & 0.04 \\ 0.08 & 0.004 & 0.08 & 0 & 0.08 \\ 0.16 & 0.008 & 0.4 & 0.08 & 0 \end{bmatrix},$$

$$\mathbf{E}_1 = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0.03 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix}. \tag{5.6}$$

Then for $\mathbf{D}_1 = \mathbf{E}_1 / \rho(\mathbf{E}_1\hat{\mathbf{Z}})$ and $\mathbf{D}_2 = \mathbf{E}_2 / \rho(\mathbf{E}_2\hat{\mathbf{Z}})$ the real roots of

$$p_2(x, y) = \det(y(\mathbf{D}_2\hat{\mathbf{Z}} - \mathbf{I}_5) + \mathbf{D}_1\hat{\mathbf{Z}} - x\mathbf{I}_5) \tag{5.7}$$

is shown in Figure 5. On close examination, one can see the line joining the corresponding feasible points on the boundary is divided into three segments. The middle segment is outside the feasible region while the remaining two segments are inside the feasible region. So neither concavity nor convexity holds even for the lining joining these two points.

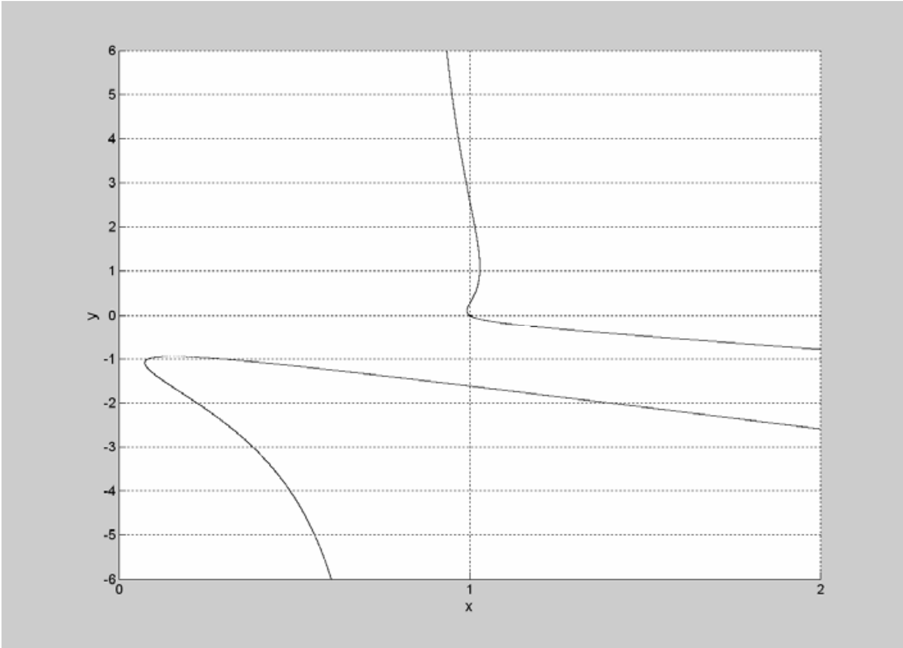

Figure 5: An example showing the complexity of the feasible region geometry

6. Conclusion

In this paper, concavity results of the feasible signal-to-noise regions are established. The results shed light on the complexity of the geometric properties of these feasible regions. There are several possible directions for future extension. In particular, in Theorem 3.4, the symmetry and the non-singular assumptions may not be necessary.

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References

[1] M. Sunay and P. McLane, "Calculating Error Probabilities for DS-CDMA Systems: When Not to Use the Gaussian Approximation," in *IEEE GLOBECOM 96*, London, UK, 1996.

[2] Y. Chen and W. S. Wong, "Power Control for non-Gaussian Interference," in *WiOpt'09*, Seoul Korea, to appear in 2009.

[3] C. W. Sung, "Log-Convexity Property of the Feasible SIR Region in Power-Controlled Cellular Systems," *IEEE Communications Letters,* vol. 6, pp. 248-249, 2002.

[4] D. Catrein, L. A. Imhof, and R. Mathar, "Power Control, Capacity, and Duality

of Uplink and Downlink in Cellular CDMA Systems " *IEEE Transactions on Communications,* vol. 52, Oct. 2004.

[5] H. Boche and S. Stańczak, "Convexity of Some Feasible QoS Regions and Asymptotic Behavior of the Minimum Total Power in CDMA Systems," *IEEE Transactions on Communications,* vol. 52, pp. 2190-2197, Dec. 2004.

[6] H. Boche and S. Stańczak, "Log-Convexity of the Minimum Total Power in CDMA Systems With Certain Quality-of-Service Guaranteed," *IEEE Transactions on Information Theory,* vol. 51, pp. 374-381, Jan. 2005.

[7] L. A. Imhof and R. Mathar, "The Geometry of the Capacity Region for CDMA Systems with General Power Constraints," *IEEE Transactions on Wireless Communications,* vol. 4, pp. 2040-2044, Sept. 2005.

[8] L. A. Imhof and R. Mathar, "Capacity Regions and Optimal Power Allocation for CDMA Cellular Radio," *IEEE Transactions on Information Theory,* vol. 51, June 2005.

[9] S. Stańczak and H. Boche, "The Infeasible SIR Region Is Not a Convex Set," *IEEE Transactions on Communications,* vol. 54, pp. 1905-1907, 2006.

[10] H. F. Weinberger, "Remarks on the Preceding of Lax," *Communications on Pure and Applied Mathematics,* vol. 11, pp. 195-196, 1958.

[11] P. D. Lax, "Differential Equations, Difference Equations and Matrix Theory," *Communications on Pure and Applied Mathematics,* vol. 11, pp. 175-194, 1958.

[12] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York: Academic Press, 1994.

[13] F. R. Gantmacher, *Matrix Theory* vol. *I*: The American Mathematical Society, 2000.