Joint Target Realization for Linear Systems Allowing Choice-Based Actions

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Abstract
Systems allowing choice-based actions are distributed control systems in which agents cooperate to achieve target states that are based on the choices selected by the agents. These choices are selected at system start time and are initially unknown to the other agents. This relatively new model is the subject of investigation in several recent papers that focus on issues such as control communication complexity, control cost and realization of target functions. This paper continues this line of research by developing optimal control design methodology for linear systems that are jointly manipulated by multiple agents. For target tensors satisfying particular structural constraints, also known as compatibility condition, we derive a set of controls that can be used to achieve chosen targets with minimum average control cost.

Key words: Choice-based actions; multi-agent systems; network control systems; distributed optimal control.

1 Introduction
Direct communication can serve an invaluable role in the instrumentation of optimal control in distributed cooperative control systems. However, efficient direct communication may not always exist. For an intuitive example, note that when movers cooperate to move a heavy object they usually “guess” the intention of their partners by “sensing” the object motion through their hands and bodies instead of using verbal communication. Signaling through motion or dynamical states were studied in recent papers on motion-based communication such as in dancing [2] [5]. Vehicle routing problems arising from fulfilling service request [1] and from traffic control [10] have been reported as solvable without explicit communication. In state stabilization of many decentralized control systems [3] [14] no central coordination nor direct information-exchange among controllers are needed. Such distributed control systems without direct communication are being investigated in this paper.

Consider a dynamic system jointly controlled by L agents, labeled from 1 to L:

\[ \dot{x}(t) = f(x(t), u_1(t, C_1), ..., u_L(t, C_L)), x(t_0) = x_0, \]

where \( u_l \) is the control input of agent l selected from a pre-determined set of control laws based on a choice parameter, \( C_l \). The choice is chosen uniformly from \{1, ..., N\} at \( t_0 \), the system initial time, and is independent of other agents’ choices. It is also only known to agent \( l \) at \( t_0 \). The objective is to find control protocols (control and communication strategies) such that given targets based on \( C_l \)’s can be realized at finite time \( t_f \). We call such a system an choice-based action system and refer to the corresponding problem as the choice-based target realization (CBTR) problem. For illustration, consider a sensor positioning system in Fig. 1 where two agents, Alice and Bob, are served by a satellite. Each agent independently selects a region of interest to monitor from a pre-defined pool. Based on the choices selected by the agents, there is a preferred position of the sensor. This problem aims to find the choice-based controls that allow the agents to jointly steer the sensor to the preferred position according to their choices.

This type of CBTR problem was first introduced in [11] for information-based systems to determine the amount of information exchange in control solutions. In [12], the authors also analyzed the inherent value of information...
exchange in comparison to the control cost for a class of nonlinear systems.

If a central coordinator exists or there are direct communication channels for the agents to exchange their choices, the CBTR problem will be reduced to a classical choice-free target-realizing problem and thus will not be a novel control problem. In the absence of both, the target feasibility question and the computation of the optimal cost of the target realizing controls become nontrivial. It is shown in [13] that for a large class of systems characterized by a bilinear input-output mapping, the existence of CBTR control depends on the rank of the mapping and the control cost without communication is much higher than that for choice-free systems. The above works differ from those in multi-agent literatures dealing with control optimization, such as [4][9] or team optimization [6], in that the system targets are non-unique and depends on parameters, the choices, that are only partially known to each agent.

In this paper, we further this line of investigation by concentrating on linear dynamical systems controlled by multiple agents. Our contributions consist of deriving explicit conditions for the CBTR problem to be feasible without direct communication and establishing design methodology for the corresponding optimal control laws. The system targets in a CBTR problem are conveniently summarized by a target tensor. A basic result resolves the question whether a CBTR problem can be solved without direct communication in terms of structural conditions on the target tensor. Target tensors satisfying the conditions are referred to as compatible, for which target-realizing linear controls that minimize an average quadratic cost function are derived. The optimal choice-based controls are given in a basic open-loop form, while suboptimal closed-loop nonlinear solutions exist and have been reported in [7].

This paper is organized into four sections including the Introduction. Section 2 states the problem formulation. In Section 3, compatibility conditions for the target tensors to be realized without communications are presented and the optimal choice-based control law is derived. A simulation example validates our results. Concluding remarks are provided in Section 4.

2 Problem formulation

Consider a jointly controlled linear system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{L} B_i u_i(t), \quad x(t_0) = x_0, \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m_i}$, $x(t) \in \mathbb{R}^n$ is the state, and $u_i(t) \in \mathbb{R}^{m_i}$, the control of agent $l$, is selected from $\{u_i^j(t), \ i = 1, 2, \ldots, N_i\}$ solely according to the choice, indexed by $i$, made by the agent. Although decentralized control systems are in general subject to a jointly controllable assumption [3], [8], this paper focuses on individually controllable systems so that basic concepts of the choice-based systems will not be cluttered by complicated formulas. Results in this paper can be extended to jointly controllable systems and will be discussed elsewhere.

Assumption 1 System (2) is controllable by each agent, i.e., for $l = 1, 2, \ldots, L$, $(A, B_l)$ are controllable pairs, or all the Grammians $W_l = \int_{t_0}^{t_f} e^{-At} B_l^T B_l e^{-A^T t} dt$ are invertible.

For the sensor positioning example cited earlier, this is a reasonable assumption to impose. Since the target to be reached depends on the choice-based actions of all agents, it is convenient to denote the intended target state when agent $l$ selects the choice $i_l$ for $l = 1, 2, \ldots, L$, by $H_{i_1i_2\ldots i_L}$. The target tensor, $H$, contains all the $N_1 \times N_2 \times \cdots \times N_L$ entries of these targets.

Although the choice of an agent may change, we assume that they are fixed in the time interval $[t_0, t_f]$. We further assume that all elements in an agent’s choice set have equal probability to be selected by that agent at time $t_0$. Under these assumptions, the objective of CBTR problem is: For an arbitrary choice combination $(i_1, i_2, \ldots, i_L)$ at $t_0$, design control strategy $u_i^j(t)$, for every agent $l \in \{1, 2, \ldots, L\}$, such that the corresponding target state $H_{i_1i_2\ldots i_L}$ is achieved, i.e,

$$x(t_f, u_1^{i_1}, u_2^{i_2}, \ldots, u_L^{i_L}) = H_{i_1i_2\ldots i_L}. \quad (3)$$

Remark 2 At the initial time, every agent will be informed of the initial system state and the target tensor $H$. Nevertheless, this CBTR problem is still a distributed one, as the choices are made by the agents independently; no agent knows a priori the exact target state $H_{i_1i_2\ldots i_L}$ to be achieved. However, the problem can be easily reduced to a centralized one if the choices of the agents are fully disclosed to each other.

Given a target tensor, we are interested in minimizing the following cost function

Fig. 1. Sensor positioning system.
\[ J = \frac{1}{\prod_{i=1}^{L} N_i} \sum_{i_1=1}^{N_1} \cdots \sum_{i_L=1}^{N_L} \int_{t_0}^{t_f} (u_{i_1}^1(t))^T u_{i_1}^1(t) \, dt \]

\[ = \int_{t_0}^{t_f} \sum_{i_l=1}^{N_l} \frac{1}{N_l} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \sum_{i_L=1}^{N_L} (u_{i_1}^1(t))^T u_{i_1}^1(t) \, dt, \]

which can be interpreted as a quadratic control cost averaged over the \( N_1 \times N_2 \times \cdots \times N_L \) cases of possible choices.

### 3 Main results

The CBTR problem may be solved by utilizing multi-round distributed control protocols where each round consists of a communication phase and a control phase [11,12]. In this paper, we focus on zero-round protocols, that is, protocols without any communication phases, which turn out to be open-loop control laws as shown later on. The condition of existence of such open-loop solutions can be stated in terms of structure conditions of the target tensor. Such a condition is first stated for solutions that are protocols without any communication phases, which can be interpreted as a quadratic control cost averaging assumptions.

#### 3.1 Target tensor

It follows from the closed-form solution to equation (2) that a target state \( H_{11\ldots L} \) can be achieved at terminal time \( t_f \) if and only if there exist controls \( \{u_i^j(t) : l = 1, 2, \ldots, L\} \) such that the following equation holds

\[ H_{11\ldots L} = e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t-t)} \sum_{l=1}^{L} B_i u_i^l(t) \, dt. \]  

(5)

If (5) holds for all entries of a target tensor, one can verify by direct checking that for any integers, \( i_j, i_j' \in \{1, 2, \ldots, N_j\} \), \( i_m, i_m' \in \{1, 2, \ldots, N_m\} \), \( l, m \in \{1, 2, \ldots, L\} \), the following equation holds

\[ H_{i_1i_2 \ldots i_m \ldots i_L} - H_{i_1i_2 \ldots i_m' \ldots i_L} = H_{i_1i_2 \ldots i_m \ldots i_L} - H_{i_1i_2 \ldots i_m' \ldots i_L}. \]

(6)

A target tensor \( H \) with all its entries satisfying (6) is called a compatible target tensor, otherwise it is called an incompatible target tensor. For example, in the sensor positioning problem, if every target state is defined to be the arithmetic mean of the positions of the regions selected by the agents, the resulting target tensor is compatible. On the other hand, one obtains an incompatible target tensor if the target is defined to be the center of the smallest circle containing the selected regions, also known as the center of the minimal covering circle.

In general, there are totally \( N_1 \times N_2 \times \cdots \times N_L \) equality constraints of the form (5). But for a compatible target tensor only \( 1 + N_1 + N_2 + \cdots + N_L - L \) of them are independent as shown in the following propositions. This fact significantly reduces the complexity of problem solving.

#### Proposition 3

Define the target state set

\[ \mathcal{H} = \{H_{11 \ldots 1} \cup \{H_{i_11 \ldots 1} : i_1 = 2 \cdots N_1\} \cup \{H_{i_1i_2 \ldots 1} : i_2 = 2 \cdots N_2\} \cup \ldots \cup \{H_{i_1i_2 \ldots i_L} : i_L = 2 \cdots N_L\}. \]

If a target tensor \( H \) is compatible, then the entries \( H_{i_1i_2 \ldots i_L} \) of \( H \) are spanned by elements in \( \mathcal{H} \). Specifically,

\[ H_{i_1i_2 \ldots i_L} = H_{i_11 \ldots 1} + H_{i_1i_21 \ldots 1} + \cdots + H_{11 \ldots i_L} - (L-1)H_{11 \ldots 1}. \]

(7)

\[ \text{PROOF.} \] This can be shown by induction. It is trivial to see that (7) holds for \( i_1 \in \{1, 2, \ldots, N_1\} \) and \( H_{i_1 \ldots 1} \) for all \( i_1 = 2 \cdots N_1 \). Now, for any, \( 1 < k < L \), assume that (7) holds for all entries \( H_{i_1i_2 \ldots i_k} \) labeled by indices with \( k \) or more of them equal to 1. Consider an arbitrary entry of the target tensor labeled by indices with exactly \( k-1 \) of them equal to 1, \( H_{i_1i_2 \ldots i_k} \). Given any two integers, \( m \) and \( n \), \( 1 \leq m < n \leq L \) we can represent the tuple of indices \( i_1 \cdots i_k \) as \( S_1i_mS_2i_nS_3 \) where \( S_1 \) stands for string \( i_1 \cdots i_{m-1} \), \( S_2 \) stands for string \( i_{m+1} \cdots i_{n-1} \), and \( S_3 \) stands for string \( i_n \cdots i_L \). It follows from (6) that

\[ H_{S_1i_mS_2i_nS_3} = H_{S_1i_mS_2S_3} + H_{S_1S_2i_nS_3} - H_{S_1S_2i_nS_3}. \]

If \( i_m \neq 1 \), \( i_n \neq 1 \), and the index tuple \( S_1i_mS_2i_nS_3 \) has \( k-1 \) of them be equal to 1, then \( H_{S_1i_mS_2S_3} \) and \( H_{S_1S_2i_nS_3} \) are entries labeled by indices with exactly \( k \) components equal to 1 and \( H_{S_1S_2i_nS_3} \) with exactly \( k+1 \) components equal to 1. Hence, all these entries satisfy (7) by induction assumption. It follows then:

\[ H_{i_1i_2 \ldots i_k} = H_{S_1i_mS_2S_3} + H_{S_1S_2i_nS_3} - H_{S_1S_2i_nS_3} = H_{i_11 \ldots 1} + H_{i_1i_21 \ldots 1} + \cdots + H_{11 \ldots i_k} - (L-1)H_{11 \ldots 1}. \]

Hence, all entries of the target tensor satisfy (7). \( \square \)

#### Proposition 4

Given a compatible target tensor \( H \), if there exists a set of controls \( \{u_i^j(t) : l = 1, 2, \ldots, L\} \) such that equation (5) is satisfied for all elements in \( \mathcal{H} \), then equation (5) holds for all entries of the \( H \).

\[ \text{PROOF.} \] According to Proposition 3, all entries of a compatible \( H \) satisfy equation (7). The proposition follows by substituting right-hand-side entries of (7) by equation (5). \( \square \)
Under Assumption 1, one can easily find linear controls to realize all elements in $\mathcal{H}$ without any communication. Combining Proposition 3, the following result follows.

**Theorem 5** Under Assumption 1, there exist linear control sets \{u_{li}^i(t) : i_l = 1, 2, \ldots, N_i, i \in \{1, 2, \ldots, L\}\}, such that all entries of a target tensor $H$ can be achieved if and only if $H$ is compatible.

In the rest of the paper, we concentrate on control laws for compatible target tensors. For incompatible tensors, results are different and will be discussed in the future.

### 3.2 Optimal controllers for compatible target tensors

In this section we present a two-step proof to the optimal CBTR problem: In the first step we derive for each agent the class of admissible control inputs that satisfy the necessary conditions for optimality irrespective of the choices selected by the other agents. In the second step, the problem is reduced to an optimization problem over an $n(L-1)$-dimensional Euclidean space for which there is a unique critical point. This critical point is then shown to be the global minimum.

**Theorem 6** Consider system (2) under Assumption 1. The set of controls that can realize all target states of a compatible target tensor $H$ while minimizing cost function (4) is defined as follows: For $i_l = 1, 2, \ldots, N_i$,

$$u_{li}^i(t) = \begin{cases} \frac{B_l^T e^{-A_t t} [W_l^{-1} e^{-A_t t} H_{1i_l \ldots -1} - e^{-A_0 x_0}]}{1 + \sum_{k=2}^{L} W_k P_k^1], l = 1} \\ \frac{B_l^T e^{-A_t t} [W_l^{-1} e^{-A_t t} (H_{1i_l \ldots -1} - H_{11 \ldots -1}) + P_k^1], l = 2, \ldots, L} \end{cases}$$

where

$$[(\tilde{P}_2^1)^T, (\tilde{P}_3^1)^T, \ldots, (\tilde{P}_L^1)^T]^T = \Omega^{-1} \Theta,$$

with

$$\Omega = \begin{bmatrix} I + W_1^{-1} W_2 & W_1^{-1} W_3 & \cdots & W_1^{-1} W_L \\ W_1^{-1} W_2 & I + W_1^{-1} W_3 & \cdots & W_1^{-1} W_L \\ \vdots & \vdots & \ddots & \vdots \\ W_1^{-1} W_2 & W_1^{-1} W_3 & \cdots & I + W_1^{-1} W_L \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} W_2^{-1} e^{-A_t t} \sum_{i_l=1}^{N_2} H_{1i_l \ldots -1 -1} N_2^{-1} e^{-A_0 x_0} \\ + W_1^{-1} \sum_{i_l=1}^{N_1} e^{-A_t t} H_{1i_l \ldots -1} N_1^{-1} e^{-A_0 x_0} \\ \vdots \\ W_L^{-1} e^{-A_t t} \sum_{i_l=1}^{N_L} H_{1i_l \ldots -1 -1} N_L^{-1} e^{-A_0 x_0} \\ + W_1^{-1} \sum_{i_l=1}^{N_1} e^{-A_t t} H_{1i_l \ldots -1} N_1^{-1} e^{-A_0 x_0} \end{bmatrix}. \tag{10}$$

**Proof.** To find the controls that minimize cost function $J$, adjoint all target constraints (5) to $J$ with real-valued Lagrange multipliers $\lambda_{1i_2 \ldots i_L} \in R^n$ and re-arrange terms

$$J = \int_0^{t_f} \sum_{i_l=1}^{N_l} \frac{1}{N_l} \sum_{i=1}^{N_i} (u_{li}^i(t))^T u_{li}^i(t) dt$$

$$+ \sum_{i=1}^{N_i} \sum_{i_l=1}^{N_l} \sum_{i_{l-1}=1}^{N_{l-1}} \sum_{i_1=1}^{N_1} \left[ X_{i_1i_2 \ldots i_L} (H_{i_1i_2 \ldots i_L} - e^{A(t_f-t_0)} x_0 \right.$$ 

$$- \int_{t_0}^{t_f} e^{A(t_f-t)} \sum_{l=1}^{L} B_l u_{li}^i(t) dt \left.] \right].$$

Now we can use the standard Lagrange method and the fundamental approach to calculus of variations to find the solution to this optimization problem. Consider the variation in $J$ due to variations in the control vectors,

$$\delta J = \sum_{l=1}^{L} \left[ \int_{t_0}^{t_f} \left( \frac{2}{N_l} (u_{li}^i(t))^T - \sum_{i=1}^{N_i} \sum_{i_{l-1}=1}^{N_{l-1}} \sum_{i_1=1}^{N_1} \sum_{l=1}^{L} \lambda_{1i_2 \ldots i_L} e^{A(t_f-t)} B_l \right) \delta u_{li}^i(t) dt \right].$$

Here we set $\delta x_0 = 0$ and $\delta H_{i_1i_2 \ldots i_L} = 0$, for $i_l = 1, 2, \ldots, N_i$, $i = 1, 2, \ldots, L$, since the initial state and the terminal states are specified. For an extremum, $\delta J$ must be zero for arbitrary $\delta u_{li}^i(t), l = 1, 2, \ldots, L$; this can happen only if the stationary solutions to $u_{li}^i(t)$’s are of the following type

$$u_{li}^i(t) = B_l^T e^{-A_t t} P_{li}^i, \tag{11}$$

where

$$P_{li}^i = \frac{N_l}{2} e^{A_t t_f} \sum_{i_{l-1}=1}^{N_{l-1}} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \lambda_{1i_2 \ldots i_L}.$$
assumes the form

$$H_{t_{1}i_{2}...i_{L}} = e^{At_{j}} \left( e^{-At_{0}}x_{0} + \sum_{i=1}^{L} W_{i}P_{i}^{1} \right). \quad (13)$$

According to Proposition 4, if equation (13) holds for all elements in \( \mathcal{H} \), it holds for an arbitrary \( H_{t_{1}i_{2}...i_{L}} \) of \( \mathcal{H} \). As a result, it is sufficient to summarize all constraints of the form (13) by the following set of equalities:

$$\begin{align*}
&\left\{ e^{-At_{j}}H_{11...1} - e^{-At_{0}}x_{0} = \sum_{k=1}^{L} W_{k}P_{k}^{1}, \\
&\quad e^{-At_{j}}H_{1i_{2}...i_{L-1}} - e^{-At_{0}}x_{0} = W_{i}P_{i}^{1} + \sum_{k=1, k \neq i}^{L} W_{k}P_{k}^{1}, \\
&\quad \text{for } i_{1} = 2, \ldots, N_{i}, l = 1, 2, \ldots, L.
\end{align*}$$

Note that (14) consists of \( N_{1} + \cdots + N_{L} + 1 - L \) equations with \( N_{1} + \cdots + N_{L} \) variables, \( \{P_{1}^{1}, l = 1, 2, \ldots, L \} \cup \{P_{i}^{l}, i = 2, \ldots, N_{i}, l = 1, 2, \ldots, L \} \). This implies there are only \( L - 1 \) free variables. Choose \( \{P_{1}^{1}, P_{2}^{1}, \ldots, P_{1}^{L} \} \) as the set of free variables. Then the remaining variables can be represented by \( \{P_{2}^{1}, P_{3}^{1}, \ldots, P_{L}^{1} \} \) as in the following equations:

$$\begin{align*}
P_{1}^{1} &= W_{1}^{-1}(e^{-At_{j}}H_{11...1} - e^{-At_{0}}x_{0} - \sum_{k=2}^{L} W_{k}P_{k}^{1}), \\
P_{l}^{i} &= \begin{cases} 
W_{1}^{-1}(e^{-At_{j}}H_{1i_{2}...i_{L}} - e^{-At_{0}}x_{0} - \sum_{k=2}^{L} W_{k}P_{k}^{1}), & l = 1 \\
W_{l}^{-1}(e^{-At_{j}}(H_{1i_{2}...i_{L-1}} - H_{11...1}) + P_{1}^{1}), & l \geq 2
\end{cases}
\end{align*}$$

Now, combining (11) and (16) with the cost function (12) leads to an equivalent problem of finding \( P^{1} = (P_{2}^{1})^{T}, (P_{3}^{1})^{T}, \ldots, (P_{L}^{1})^{T} \)^{T}, such that the following function is minimized

$$J(P^{1}) = J(P_{1}^{1}, P_{2}^{1}, \ldots, P_{L}^{1}) \geq \frac{1}{N_{i}} \sum_{i=1}^{N_{1}} \left| W_{1}^{-1}(e^{-At_{j}}H_{1i_{2}...i_{L}} - e^{-At_{0}}x_{0} - \sum_{k=2}^{L} W_{k}P_{k}^{1}) \right|^{T} \cdot W_{1}[W_{1}^{-1}(e^{-At_{j}}H_{1i_{2}...i_{L}} - e^{-At_{0}}x_{0} - \sum_{k=2}^{L} W_{k}P_{k}^{1})]^{T}$$

$$+ \sum_{l=2}^{L} \sum_{i=1}^{N_{i}} \left| W_{l}^{-1}(e^{-At_{j}}(H_{1i_{2}...i_{L-1}} - H_{11...1}) + P_{1}^{1}) \right|^{T} \cdot W_{l}[W_{l}^{-1}(e^{-At_{j}}(H_{1i_{2}...i_{L-1}} - H_{11...1}) + P_{1}^{1})].$$

(17)

By means of the equations \( \partial J(P^{1}) / \partial P_{l}^{1} = 0 \) for all \( l \geq 2 \), one can show that a critical point for this problem must be given by \( P_{l}^{1} = P_{l}^{1} \), for all \( l \geq 2 \), as defined in (9). Moreover, this solution is uniquely determined by the system parameters. We next prove that (9) defines the global minimum of (17). Denoting the minimum of all the eigenvalues of all the \( W_{i} \)'s by \( \alpha \), which is positive since all \( W_{i} \)'s are positive definite, we can have

$$J(P^{1}) = J(P_{2}^{1}, P_{3}^{1}, \ldots, P_{L}^{1}) \geq \sum_{l=2}^{L} \frac{1}{N_{i}} \sum_{i=1}^{N_{i}} \alpha \| W_{l}^{-1}(H_{1i_{2}...i_{L}} - H_{11...1}) + P_{l}^{1} \|^{2}$$

$$= \alpha \sum_{l=2}^{L} \sum_{i=1}^{N_{i}} \left( \sum_{i=1}^{N_{i}} \| D_{i} + P_{l}^{1} \| \right)^{2}$$

$$\geq \alpha \sum_{l=2}^{L} \sum_{i=1}^{N_{i}} \left( \| P_{l}^{1} \| - \| D_{i} \| \right)^{2}$$

$$\geq \alpha \sum_{l=2}^{L} \left( \| P_{l}^{1} \| - \| D_{l} \| \right)^{2}$$

$$= \alpha \sum_{l=2}^{L} \left( \| P_{l}^{1} \| - \| D_{l} \| \right)^{2}$$

where \( D_{i} = W_{l}^{-1}(H_{1i_{2}...i_{L}} - H_{11...1}) \) and \( \| D_{l} \| = \max_{i \in \{1, 2, \ldots, N_{l}\}} \| D_{i} \| \). For an arbitrary number \( C_{P} > 0 \), on the circle \( \sum_{l=2}^{L} \| P_{l}^{1} \|^{2} = C_{P}^{2} \), the inequality

$$\sum_{l=2}^{L} \| P_{l}^{1} \|^{2} \geq \frac{1}{L - 1} \left( \sum_{l=2}^{L} \| P_{l}^{1} \| \right)^{2}$$

implies \( \sum_{l=2}^{L} \| P_{l}^{1} \| \leq \sqrt{L - 1} C_{P} \). Therefore, we have

$$J(P^{1}) \geq \alpha(C_{P}^{2} - \sum_{l=2}^{L} \| P_{l}^{1} \| \| D_{l} \| + D^{2})$$

$$\geq \alpha(C_{P}^{2} - 2d\sqrt{L - 1} C_{P} + D^{2})$$

where \( D = \sum_{l=2}^{L} \| D_{l} \|^{2} \) and \( d = \max_{i \in \{1, 2, \ldots, L\}} \| D_{i} \| \). Since \( D \) and \( d \) are constants about parameters \( A, B_{l}, x_{0}, \) and \( t_{j} \), we can guarantee \( J(P^{1}) > J(P_{2}^{1}, P_{3}^{1}, \ldots, P_{L}^{1}) \) if \( C_{P} \) is sufficiently large. This indicates that the minimum point cannot lie on the boundary of the bounded domain defined by \( \sum_{l=2}^{L} \| P_{l}^{1} \|^{2} \leq C_{P}^{2} \). Moreover, \( J(\cdot) \) is a continuously differentiable function in the region \( \sum_{l=2}^{L} \| P_{l}^{1} \|^{2} \leq C_{P}^{2} \) and must have a global minimum. So the global minimum is an interior point and hence must satisfy the first order necessary condition. As claimed before, (9) is the unique critical point derived from the first order necessary condition, therefore, it must be the global minimum of (17). By setting \( P_{l}^{1} = P_{l}^{1} \), for all \( l \geq 2 \), we can specify all the \( P_{l}^{1} \) via (16). Finally, defining the controls via (11), we obtain the set of optimal solutions as given in Theorem 6. This completes the proof. \( \square \)
3.3 A Numerical Example

Consider a sensor positioning system involving three agents $A$, $B$ and $C$, each of whom has two regions marked in X-Y coordinates in Fig. 2. The sensor’s position $(x, y)$ is described by equation (2) with $A = \begin{bmatrix} A_x & 0 \\ 0 & A_y \end{bmatrix}$, $A_x = A_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_l = \begin{bmatrix} B_x & 0 \\ 0 & B_y \end{bmatrix}$, $B_x = B_y = 0$, $l = 1, 2, 3$. The target position is defined as the arithmetic mean of the selected regions (see Fig. 2), and hence the target tensor $H$ is compatible.

By applying the controls in Theorem 6, numerical results in Fig. 2 show that starting from $x(t_0)$, the satellite reaches the target position $H_{111}$ if the choice combination of the three agents is $(t_1 = 1, t_2 = 1, t_3 = 1)$, and reaches $H_{122}$ if the choice combination is $(t_1 = 1, t_2 = 2, t_3 = 2)$ at time $t_1 > t_0$. In successive time periods $[t_1, t_2], [t_2, t_3] \cdots$, the satellite transfers among the target positions according to choice changes of any agent. This example verifies the effectiveness of our control law.

Table 1

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<th>$H_{111}$</th>
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<td>2600</td>
<td>3600</td>
<td>\cdots</td>
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<td>4400</td>
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<td>CF</td>
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</table>

4 Concluding Remarks

In the present paper we investigated the design problem of linear choice-based action systems in which multiple agents apply controls based on their independent choices. A systematic design methodology for optimal control of general multi-agent linear systems has been established for target tensors satisfying the compatibility conditions. Solution methodology for incompatible target tensors is reported elsewhere. The derived open-loop control law does not require any communication to indicate the choices selected by the agent, however, it tends to incur higher control cost than cases where the choices are a priori fixed as shown in Table 1. To reduce control cost, some level of communication to indicate the choices of agents in a control protocol will help as shown in [13] for bilinear systems. Therefore, the control communication complexity problem should also be analyzed for such linear systems.

References


