

Joint Target Realization for Linear Systems Allowing Choice-Based Actions[☆]

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Abstract

Systems allowing choice-based actions are distributed control systems in which multiple agents cooperate to achieve system targets that are determined by the choices of the agents. The choice of every agent is undisclosed to the other agents and thus brings in uncertainties to each other. In the absence of direct inter-agent communication, the feasibility of control laws for a class of systems governed by linear dynamics is investigated. Optimal control laws are designed such that some given set of *compatible* targets can all be achieved.

Keywords: Choice-based actions, Joint target realization, Distributed optimal control

1. Introduction

Direct communication can serve an invaluable role in the instrumentation of control in distributed systems cooperatively controlled by multiple agents. However, efficient direct communication may not always exist. For an intuitive example, note that when movers cooperate to move a heavy object they usually “guess” the intention of their partners by “sensing” the object motion through their hands and bodies instead of using verbal communication. Without explicit communication, vehicle routing problems arising from fulfilling service request [1] and from traffic control [2] have been reported as solvable. Interactive tasks such as dance is still possible by signaling through motion or dynamical states [3] [4]. Such distributed control systems without direct communication among agents are being investigated in this paper.

Agents sharing a common system dynamics may have individual options and they make choices independent of each other. For illustration, consider a sensor positioning system in Fig. 1 where two agents, Alice and Bob, are served by a satellite. Each agent independently selects a region of interest to monitor from their individual pool at initial time. Based

[☆]This work is supported by the National Natural Science Foundation of China (61174060), and Shenzhen matching grant (GJHS 20120702105523301).

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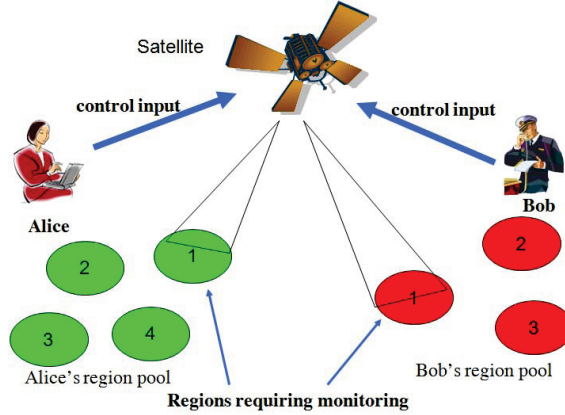


Figure 1: Choice-based sensor positioning system.

on the regions selected by the two agents, there is a preferred position of the sensor and this target position is not known to either agent. The question is whether one can find controls that enables the agents to jointly steer the sensor to the preferred position. Such a system is called a *choice-based action system* and the problem is referred to as the *choice-based target realization* (CBTR) problem. If a central coordinator exists or there are direct communication channels for the agents to exchange their choices, this CBTR problem will be reduced to a classical choice-free target-realizing problem and thus trivializes. In the absence of both, the target feasibility question and the computation of the optimal cost of the target realizing controls become nontrivial.

This type of distributed control system was initially introduced in [5, 6] for the purpose of determining the amount of information exchange in control solutions. In [7], a class of systems characterized by a bilinear input-output mapping was considered, and it is shown that the existence of CBTR control for such systems depends on the rank of the mapping. This line of works differ from those in multi-agent literatures dealing with control optimization [8] or team optimization [9], in that the system targets are non-unique and depends on parameters, the choices, that are only partially known to each agent.

In this paper, we concentrate on systems governed by linear dynamics. Unlike the previous works where only two controllers are involved, we allow multiple control agents in the system. Our results consist of deriving explicit conditions for the CBTR problem to be feasible without direct communication and establishing design methodology for the corresponding optimal control laws.

This paper is organized into four sections including the Introduction. Section 2 states the problem formulation. In Section 3, compatibility conditions for all targets to be realized without communications are presented and the optimal choice-based control law is derived. A simulation example validates our results. Concluding remarks are provided in Section 4.

2. Problem formulation

Consider a jointly controlled linear system

$$\dot{x}(t) = Ax(t) + \sum_{l=1}^L B_l u_l(t), \quad x(t_0) = x_0, \quad (1)$$

where $x(t) \in R^n$ is the system state, $A \in R^{n \times n}$, $B_l \in R^{n \times m_l}$, and $u_l(t) \in R^{m_l}$, the control of agent l , is selected from $\{u_l^{i_l}(t), i_l = 1, 2, \dots, N_l\}$ solely according to the choice, indexed by i_l , made by the agent. We assume that system (1) is individually controllable by each agent.

Assumption 1. *System (1) is controllable by each agent, i.e., for $l = 1, \dots, L$, (A, B_l) are controllable pairs, or all the controllability Grammians $W_l = \int_{t_0}^{t_f} e^{-At} B_l B_l^T e^{-A^T t} dt$ are invertible.*

For the sensor positioning example cited earlier, this is a reasonable assumption to impose. For systems that are only jointly controllable, i.e., $(A, [B_1, B_2 \dots B_L])$ is controllable, one can decompose the state space into subspaces that are controllable by that subset of agents [10][11]. If there exist subspaces that are controllable by only one agent, then the subproblem trivializes. Therefore, it is more meaningful as well as concise to make this assumption.

Since the target to be reached depends on the choice-based actions of all agents, it is convenient to denote the intended target state when agent l selects the choice i_l , for $l = 1, 2, \dots, L$, by $H_{i_1 i_2 \dots i_L}$. Then, the *target tensor*, $H = [H_{i_1 i_2 \dots i_L}]$, contains all these $N_1 \times N_2 \times \dots \times N_L$ targets. These targets represent different objectives of the system corresponding to the choice outcomes of the agents and are assumed pre-defined. We are dedicated to the control synthesis problem for a given target tensor, while the derivation of proper target values for choice actions are beyond the scope of this paper.

The choice outcomes of all agents are assumed fixed in the time interval $[t_0, t_f]$, and all elements in an agent's choice set have an equal probability to be selected by that agent at time t_0 . Under these assumptions, the objective of CBTR problem is: For an arbitrary choice combination $(i_1, i_2, \dots, i_L)$ at t_0 , design control strategy $u_l^{i_l}(t)$, for every agent $l \in \{1, 2, \dots, L\}$, such that the corresponding target state $H_{i_1 i_2 \dots i_L}$ is achieved, i.e,

$$x(t_f, u_1^{i_1}, u_2^{i_2}, \dots, u_L^{i_L}) = H_{i_1 i_2 \dots i_L}. \quad (2)$$

The challenge of this problem is that no agent will know a priori the exact target state $H_{i_1 i_2 \dots i_L}$ to be achieved, since the choices are made by the agents independently and are not disclosed to each other.

As in [7], we are also desired to minimize the following cost function

$$J = \frac{1}{\prod_{l=1}^L N_l} \sum_{i_1=1}^{N_1} \dots \sum_{i_L=1}^{N_L} \int_{t_0}^{t_f} (u_l^{i_l}(t))^T u_l^{i_l}(t) dt = \int_{t_0}^{t_f} \sum_{l=1}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} (u_l^{i_l}(t))^T u_l^{i_l}(t) dt, \quad (3)$$

which can be interpreted as a quadratic control cost averaged over the $N_1 \times N_2 \times \dots \times N_L$ cases of possible choices.

3. Main results

The CBTR problem may be solved by utilizing multi-round distributed control protocols where each round consists of a communication phase and a control phase [7]. In this paper, we focus on zero-round protocols, that is, protocols without any communication phases, which turn out to be open-loop control laws as shown later on. The condition of existence of such open-loop solutions can be stated in terms of structure conditions of the target tensor.

3.1. Target tensor

It follows from the closed-form solution to equation (1) that a target state $H_{i_1 i_2 \dots i_L}$ can be achieved at terminal time t_f if and only if there exist controls $\{u_l^{i_l}(t), l = 1, 2, \dots, L\}$ such that the following equation holds

$$H_{i_1 \dots i_L} = e^{A(t_f - t_0)} x_0 + \int_{t_0}^{t_f} e^{A(t_f - t)} \sum_{l=1}^L B_l u_l^{i_l}(t) dt. \quad (4)$$

If (4) holds for all entries of a target tensor, one can verify by direct checking that for any integers, $i_l, i'_l \in \{1, 2, \dots, N_l\}$, $i_m, i'_m \in \{1, 2, \dots, N_m\}$, $l, m \in \{1, 2, \dots, L\}$, the following equation holds

$$H_{i_1 i_2 \dots i_l \dots i_m \dots i_L} - H_{i_1 i_2 \dots i_l \dots i'_m \dots i_L} = H_{i_1 i_2 \dots i'_l \dots i_m \dots i_L} - H_{i_1, i_2 \dots i'_l \dots i'_m \dots i_L}. \quad (5)$$

A target tensor H with all its entries satisfying (5) is called a *compatible target tensor*, otherwise it is called an *incompatible target tensor*. For example, in the sensor positioning problem, if every target state is defined to be the arithmetic mean of the positions of the regions selected by the agents, the resulting target tensor is compatible. On the other hand, one obtains an incompatible target tensor if the target is defined to be the center of the smallest circle containing the selected regions, also known as the center of the minimal covering circle.

In general, there are totally $N_1 \times N_2 \times \dots \times N_L$ equality constraints of the form (4). But for a compatible target tensor only $1 + N_1 + N_2 + \dots + N_L - L$ of them are independent as shown in the following propositions. This fact significantly reduces the complexity of problem solving.

Proposition 1. *Define a collection of target states*

$$\begin{aligned} \mathcal{H} = & \{H_{11 \dots 1}\} \cup \{H_{i_1 1 \dots 1}, i_1 = 2 \dots N_1\} \cup \{H_{1 i_2 \dots 1}, i_2 = 2 \dots N_2\} \cup \dots \\ & \cup \{H_{11 \dots i_L}, i_L = 2 \dots N_L\}. \end{aligned}$$

If a target tensor H is compatible, then the entries $H_{i_1 i_2 \dots i_L}$ of H are spanned by elements in $\mathcal{H}$. Specifically,

$$H_{i_1 i_2 \dots i_L} = H_{i_1 1 \dots 1} + H_{1 i_2 1 \dots 1} + \dots + H_{1 \dots 1 i_L} - (L - 1)H_{11 \dots 1}. \quad (6)$$

Proof 1. This can be shown by induction. It is trivial to see that (6) holds for $H_{11\dots 1}$ and $H_{1\dots 1, i_l, 1\dots 1}$ for all $i_l = 2 \dots N_l$. Now, for any, $1 < k < L$, assume that (6) holds for all entries $H_{i_1 i_2 \dots i_L}$ labeled by indices with k or more of them equal to 1. Consider an arbitrary entry of the target tensor labeled by indices with exactly $k - 1$ of them equal to 1, $H_{i_1 i_2 \dots i_L}$. Given any two integers, m and n , $1 \leq m < n \leq L$ we can represent the tuple of indices $i_1 i_2 \dots i_L$ as $S_1 i_m S_2 i_n S_3$ where S_1 stands for string $i_1 \dots i_{m-1}$, S_2 stands for string $i_{m+1} \dots i_{n-1}$, and S_3 stands for string $i_{n+1} \dots i_L$. It follows from (5) that

$$H_{S_1 i_m S_2 i_n S_3} = H_{S_1 i_m S_2 1 S_3} + H_{S_1 1 S_2 i_n S_3} - H_{S_1 1 S_2 1 S_3}.$$

If $i_m \neq 1$, $i_n \neq 1$, and the index tuple $S_1 i_m S_2 i_n S_3$ has $k - 1$ of them be equal to 1, then $H_{S_1 i_m S_2 1 S_3}$, $H_{S_1 1 S_2 i_n S_3}$ are entries labeled by indices with exactly k components equal to 1 and $H_{S_1 1 S_2 1 S_3}$ with exactly $k + 1$ components equal to 1. Hence, all these entries satisfy (6) by induction assumption. It follows then:

$$\begin{aligned} H_{i_1 i_2 \dots i_L} &= H_{S_1 i_m S_2 i_n S_3} = H_{S_1 i_m S_2 1 S_3} + H_{S_1 1 S_2 i_n S_3} - H_{S_1 1 S_2 1 S_3} \\ &= H_{i_1 1 \dots 1} + H_{1 i_2 1 \dots 1} + \dots + H_{1 \dots 1 i_L} - (L - 1)H_{1 \dots 1}. \end{aligned}$$

Hence, all entries of the target tensor satisfy (6). $\square$

Proposition 2. Given a compatible target tensor H , if there exists a set of controls $\{u_l^{i_l}(t) : i_l = 1, 2, \dots, N_l, l = 1, 2, \dots, L\}$ such that equation (4) is satisfied for all elements in $\mathcal{H}$, then equation (4) holds for all entries of H .

Proof 2. According to Proposition 1, all entries of a compatible H satisfy equation (6). The proposition follows by substituting right-hand-side entries of (6) by equation (4). $\square$

Under Assumption 1, one can easily find linear controls such that all elements in $\mathcal{H}$ can be realized without any inter-agent communication. Combining Proposition 1, the following result follows.

Theorem 1. Under Assumption 1, there exist linear control sets $\{u_l^{i_l}(t), i_l = 1, 2, \dots, N_l\}$, $l \in \{1, 2, \dots, L\}$, such that all entries of a target tensor H can be achieved if and only if H is compatible.

Now, we can design optimal control laws for compatible target tensors.

3.2. Optimal controllers for compatible target tensors

In this subsection, we present a two-step proof to the optimal CBTR problem: In the first step we derive for each agent the class of admissible control inputs that satisfy the necessary conditions for optimality irrespective of the choices selected by the other agents. In the second step, the problem is reduced to an optimization problem over an $n(L - 1)$ -dimensional Euclidean space for which there is a unique critical point. This critical point is then shown to be the global minimum.

Theorem 2. Consider system (1) under Assumption 1. The set of controls that can realize all target states of a compatible target tensor H while minimizes cost function (3) is defined as follows: For $i_l = 1, 2, \dots, N_l$,

$$\hat{u}_l^{i_l}(t) = \begin{cases} B_1^T e^{-A^T t} [W_1^{-1} (e^{-At_f} H_{i_1 1 \dots 1} - e^{-At_0} x_0 - \sum_{k=2}^L W_k \hat{P}_k^1)], & l = 1 \\ B_l^T e^{-A^T t} [W_l^{-1} e^{-At_f} (H_{1 \dots i_l \dots 1} - H_{11 \dots 1}) + \hat{P}_l^1], & 2 \leq l \leq L \end{cases} \quad (7)$$

where

$$[(\hat{P}_2^1)^T, (\hat{P}_3^1)^T, \dots, (\hat{P}_L^1)^T]^T = \Omega^{-1} \Theta, \quad (8)$$

with

$$\Omega = \begin{bmatrix} I + W_1^{-1} W_2 & W_1^{-1} W_3 & \cdots & W_1^{-1} W_L \\ W_1^{-1} W_2 & I + W_1^{-1} W_3 & \cdots & W_1^{-1} W_L \\ \vdots & \vdots & \ddots & \vdots \\ W_1^{-1} W_2 & W_1^{-1} W_3 & \cdots & I + W_1^{-1} W_L \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} W_2^{-1} e^{-At_f} \sum_{i_2=1}^{N_2} \frac{H_{11 \dots 1} - H_{1 i_2 1 \dots 1}}{N_2} \\ + W_1^{-1} \sum_{i_1=1}^{N_1} \frac{e^{-At_f} H_{11 \dots 1} - e^{-At_0} x_0}{N_1} \\ \vdots \\ W_L^{-1} e^{-At_f} \sum_{i_L=1}^{N_L} \frac{H_{11 \dots 1} - H_{11 \dots L}}{N_L} \\ + W_1^{-1} \sum_{i_1=1}^{N_1} \frac{e^{-At_f} H_{11 \dots 1} - e^{-At_0} x_0}{N_1} \end{bmatrix}. \quad (9)$$

Proof 3. To find the controls that minimize cost function J , adjoin all target constraints (4) to J with real-valued Lagrange multipliers $\lambda_{i_1 i_2 \dots i_L} \in R^n$ and re-arrange terms

$$\begin{aligned} \bar{J} = & \int_0^{t_f} \sum_{l=1}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} (u_l^{i_l}(t))^T u_l^{i_l}(t) dt + \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \cdots \\ & \sum_{i_L=1}^{N_L} \left[\lambda_{i_1 i_2 \dots i_L}^T \left(H_{i_1 i_2 \dots i_L} - e^{A(t_f - t_0)} x_0 - \int_{t_0}^{t_f} e^{A(t_f - t)} \sum_{l=1}^L B_l u_l^{i_l}(t) dt \right) \right]. \end{aligned}$$

Now we can use the standard Lagrange method and the fundamental approach to calculus of variations to find the solution to this optimization problem. Consider the variation in $\bar{J}$ due to variations in the control vectors,

$$\begin{aligned} \delta \bar{J} = & \sum_{l=1}^L \left[\int_{t_0}^{t_f} \left(\frac{2}{N_l} (u_l^{i_l}(t))^T - \sum_{i_1=1}^{N_1} \cdots \sum_{i_{l-1}=1}^{N_{l-1}} \right. \right. \\ & \left. \left. \sum_{i_{l+1}=1}^{N_{l+1}} \cdots \sum_{i_L=1}^{N_L} \lambda_{i_1 i_2 \dots i_L} e^{A(t_f - t)} B_l \right) \delta u_l^{i_l}(t) dt \right]. \end{aligned}$$

Here we set $\delta x_0 = 0$ and $\delta H_{i_1 i_2 \dots i_L} = 0$, for $i_l = 1, 2, \dots, N_l$, $l = 1, 2, \dots, L$, since the initial state and the terminal states are specified. For an extremum, $\delta \bar{J}$ must be zero for arbitrary

$\delta u_l^{i_l}(t)$, $l = 1, 2, \dots, L$; this can happen only if the stationary solutions to $u_l^{i_l}(t)$'s are of the following type

$$u_l^{i_l}(t) = B_l^T e^{-A^T t} P_l^{i_l}, \quad (10)$$

where $P_l^{i_l} = \frac{N_l}{2} e^{A^T t_f} \sum_{i_1=1}^{N_1} \dots \sum_{i_{l-1}=1}^{N_{l-1}} \sum_{i_{l+1}=1}^{N_{l+1}} \dots \sum_{i_L=1}^{N_L} \lambda_{i_1 i_2 \dots i_L}$. In addition to the necessary condition arising from zero first-order variation of $\bar{J}$, the non-negativity of the second-order variation for all values of $\delta u_l^{i_l}(t)$ is easy to verify. Therefore, among all possible types of controls the class of controls given in (10) yields the least average control cost $\bar{J}$. The only thing left to determine is that within this class of controls the Lagrange multipliers $\lambda_{i_1 i_2 \dots i_L}$, or equivalently, the set of values for $P_l^{i_l}$, such that cost function (3), or the following cost,

$$J = \sum_{l=1}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} (P_l^{i_l})^T W_l P_l^{i_l}, \quad (11)$$

is minimized. For this reduced problem, the equality constraint in (4) for all $i_l = 1, 2, \dots, N_l$, $l = 1, 2, \dots, L$, assumes the form

$$H_{i_1 i_2 \dots i_L} = e^{A t_f} \left(e^{-A t_0} x_0 + \sum_{l=1}^L W_l P_l^{i_l} \right). \quad (12)$$

According to Proposition 2, if equation (12) holds for all elements in $\mathcal{H}$, it holds for an arbitrary $H_{i_1 i_2 \dots i_L}$ of H . As a result, it is sufficient to summarize all constraints of the form (12) by the following set of equalities:

$$\begin{cases} e^{-A t_f} H_{11 \dots 1} - e^{-A t_0} x_0 = \sum_{k=1}^L W_k P_k^1, \\ e^{-A t_f} H_{1 \dots i_l \dots 1} - e^{-A t_0} x_0 = W_l P_l^{i_l} + \sum_{k=1, k \neq l}^L W_k P_k^1, \quad i_l = 2 \dots N_l, l = 1 \dots L. \end{cases} \quad (13)$$

Note that (13) consists of $N_1 + \dots + N_L + 1 - L$ equations with $N_1 + \dots + N_L$ variables, $\{P_l^1, l = 1, 2, \dots, L\} \cup \{P_l^{i_l}, i_l = 2, \dots, N_l, l = 1, 2, \dots, L\}$. This implies there are only $L - 1$ free variables. Choose $\{P_2^1, P_3^1, \dots, P_L^1\}$ as the set of free variables. Then the remaining variables can be represented by $\{P_2^1, P_3^1, \dots, P_L^1\}$ as in the following equations:

$$P_1^1 = W_1^{-1} (e^{-A t_f} H_{11 \dots 1} - e^{-A t_0} x_0 - \sum_{k=2}^L W_k P_k^1), \quad (14)$$

$$P_l^{i_l} = \begin{cases} W_1^{-1} (e^{-A t_f} H_{1 \dots i_l \dots 1} - e^{-A t_0} x_0 - \sum_{k=2}^L W_k P_k^1), & l = 1 \\ W_l^{-1} e^{-A t_f} (H_{1 \dots i_l \dots 1} - H_{11 \dots 1}) + P_l^1, & 2 \leq l \leq L \end{cases} \quad (15)$$

Now, combining (10) and (15) with the cost function (11) leads to an equivalent problem of

finding $\bar{P}^1 = [(P_2^1)^T, (P_3^1)^T, \dots, (P_L^1)^T]^T$, such that the following function is minimized

$$\begin{aligned}
J(\bar{P}^1) &= J(P_2^1, P_3^1, \dots, P_L^1) \\
&= \frac{1}{N_1} \sum_{i_1=1}^{N_1} [W_1^{-1}(e^{-At_f} H_{1\dots i_1\dots 1} - e^{-At_0} x_0 - \sum_{k=2}^L W_k P_k^1)]^T \\
&\quad \cdot W_1 [W_1^{-1}(e^{-At_f} H_{1\dots i_1\dots 1} - e^{-At_0} x_0 - \sum_{k=2}^L W_k P_k^1)] \\
&\quad + \sum_{l=2}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} [W_l^{-1} e^{-At_f} (H_{1\dots i_l\dots 1} - H_{11\dots 1}) + P_l^1]^T \\
&\quad \cdot W_l [W_l^{-1} e^{-At_f} (H_{1\dots i_l\dots 1} - H_{11\dots 1}) + P_l^1].
\end{aligned} \tag{16}$$

By means of the equations $\partial J(\bar{P}^1)/\partial P_l^1 = 0$ for all $l \geq 2$, one can show that a critical point for this problem must be given by $P_l^1 = \hat{P}_l^1$, for all $l \geq 2$, as defined in (8). Moreover, this solution is uniquely determined by the system parameters. We next prove that (8) defines the global minimum of (16). Denoting the minimum of all the eigenvalues of all the W_l 's by α , which is positive since all W_l 's are positive definite, we can have

$$\begin{aligned}
J(\bar{P}^1) &= J(P_2^1, P_3^1, \dots, P_L^1) \geq \sum_{l=2}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} \alpha \|W_l^{-1} e^{-At_f} (H_{1\dots i_l\dots 1} - H_{11\dots 1}) + P_l^1\|^2 \\
&= \alpha \sum_{l=2}^L \frac{1}{N_l} \sum_{i_l=1}^{N_l} \|D_{i_l} + P_l^1\|^2 \geq \alpha \sum_{l=2}^L \frac{1}{N_l^2} \left[\sum_{i_l=1}^{N_l} \|D_{i_l} + P_l^1\| \right]^2 \\
&\geq \alpha \sum_{l=2}^L \frac{1}{N_l^2} \left[\sum_{i_l=1}^{N_l} (\|P_l^1\| - \|D_{i_l}\|) \right]^2 \geq \alpha \sum_{l=2}^L \frac{1}{N_l^2} [N_l (\|P_l^1\| - \|D_l\|)]^2 \\
&= \alpha \sum_{l=2}^L (\|P_l^1\| - \|D_l\|)^2
\end{aligned}$$

where $D_{i_l} = W_l^{-1} e^{-At_f} (H_{1\dots i_l\dots 1} - H_{11\dots 1})$ and $\|D_l\| = \max_{i_l \in \{1, 2, \dots, N_l\}} \|D_{i_l}\|$. For an arbitrary number $C_P > 0$, on the circle $\sum_{l=2}^L \|P_l^1\|^2 = C_P^2$, the inequality $\sum_{l=2}^L \|P_l^1\|^2 \geq \frac{1}{L-1} \left(\sum_{l=2}^L \|P_l^1\| \right)^2$ implies $\sum_{l=2}^L \|P_l^1\| \leq \sqrt{L-1} C_P$. Therefore, we have

$$J(\bar{P}^1) \geq \alpha(C_P^2 - 2 \sum_{l=2}^L \|P_l^1\| \|D_l\| + D^2) \geq \alpha(C_P^2 - 2d\sqrt{L-1}C_P + D^2)$$

where $D = \sum_{l=2}^L \|D_l\|^2$ and $d = \max_{l \in \{2, \dots, L\}} \|D_l\|$. Since D and d are constants about parameters A , B_l , x_0 , H , and t_f , we can guarantee $J(\bar{P}^1) > J(\hat{P}_2^1, \hat{P}_3^1, \dots, \hat{P}_L^1)$ if C_P is sufficiently large. This indicates that the minimum point cannot lie on the boundary of the

bounded domain defined by $\sum_{l=2}^L \|P_l^1\|^2 \leq C_P^2$. Moreover, $J(\cdot)$ is a continuously differentiable function in the region $\sum_{l=2}^L \|P_l^1\|^2 \leq C_P^2$ and must have a global minimum. So the global minimum is an interior point and hence must satisfy the first order necessary condition. As claimed before, (8) is the unique critical point derived from the first order necessary condition, therefore, it must be the global minimum of (16). By setting $P_l^1 = \hat{P}_l^1$, for all $l \geq 2$, we can specify all the $P_l^{i_l}$ via (15). Finally, defining the controls via (10), we obtain the set of optimal solutions as given in Theorem 2. This completes the proof. $\square$

Remark 1. Note that each control law $u_l^{i_l}(t)$ is dependent on all the system information except the actual choices of other agents. Hence, they are still distributed control laws and can be computed by the agents separately. These controls allow the agents cooperatively drive the system state to reach the desired target without know the actually selected target in advance.

3.3. A Numerical Example

Consider a sensor positioning system involving three agents **a**, **b** and **c**, each of whom has two regions marked in X-Y coordinates in Fig. 2. The sensor's position (x, y) is described by equation (1) with $A = \begin{bmatrix} A_x & \mathbf{0} \\ \mathbf{0} & A_y \end{bmatrix}$, $A_x = A_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_l = \begin{bmatrix} B_x & \mathbf{0} \\ \mathbf{0} & B_y \end{bmatrix}$, $B_x = B_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $l = 1, 2, 3$. The target position is defined as the arithmetic mean of the selected regions (see Fig. 2), and hence the target tensor H is compatible.

By applying the controls in Theorem 2, numerical results in Fig. 2 show that starting from $x(t_0)$, the satellite reaches the target position H_{111} if the choice combination of the three agents is $(i_1 = 1, i_2 = 1, i_3 = 1)$, and reaches H_{122} if the choice combination is $(i_1 = 1, i_2 = 2, i_3 = 2)$ at time $t_1 > t_0$. In successive time periods $[t_1, t_2]$, $[t_2, t_3] \dots$, the satellite transfers among the target positions according to choice changes of any agent. This example verifies the effectiveness of our control law.

Table 1: costs of choice-based (CB) and choice-free (CF) control laws (from t_0 to t_1)

	H_{111}	H_{112}	H_{121}	$\dots$	H_{221}	H_{222}	average
CB	$\frac{2000}{3}$	$\frac{5600}{3}$	$\frac{3800}{3}$	$\dots$	$\frac{5000}{3}$	$\frac{8600}{3}$	$\frac{5300}{3}$
CF	$\frac{1400}{3}$	$\frac{4400}{3}$	$\frac{2600}{3}$	$\dots$	$\frac{3800}{3}$	$\frac{8000}{3}$	$\frac{4400}{3}$

4. Concluding Remarks

In the present paper we investigated the control problem of linear choice-based action systems involving multiple agents. Optimal open-loop controllers are derived for target tensors satisfying the compatibility conditions. For systems with noises, closed-loop solutions are desired and a preliminary suboptimal solution which can only handle the two-agent case

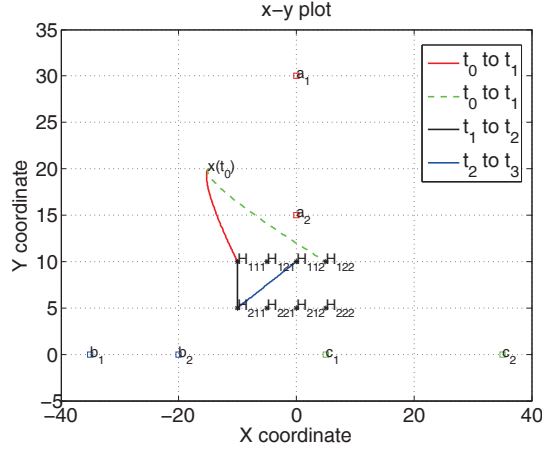


Figure 2: Simulation for the sensor positioning system with three agents **a**, **b**, **c**. $\square$ represents coordinates of the agents' regions; * represents target positions of the sensor.

was reported in [12]. More work need to be done along this line of investigation. It is noticed that control laws without direct inter-agent communication tends to incur higher control cost than cases where the choices are a priori fixed as shown in Table 1. To reduce control cost, some level of communication to indicate the choices of agents in a control protocol will help as shown in [7] for bilinear systems. Therefore, the control communication complexity problem [5, 6] should also be analyzed for such linear systems.

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