The Competition on the Mathematics of Information 2024

The Chinese University of Hong Kong

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SOLUTIONS

Question	Points
1	70
2	70
3	70
4	70
Total	280

Question 1: Quantization

In image compression, a common technique is quantization, where the value of a pixel is converted into another similar value that has a smaller set of possibilities. For simplicity, consider a grayscale image, where each pixel has a value that is an integer in the range $0, 1, \ldots, 99, 100$. To compress the value x_0 in the range $0, 1, \ldots, 100$ by a quantizer with step size 11 and initial level 0, we compress x_0 into the value x_1 among $0, 11, 22, \ldots, 99$ that is closest to x. For example, if $x_0 = 50$, it is compressed into $x_1 = 55$, which is the number divisible by 11 within the range $0, 1, \ldots, 100$ that is closest to x_0. This way, we can reduce the 101 different possible values of x_0 into only 10 different possible values of x_1, which requires a smaller space to store in a computer.

More generally, to compress x_0 by a quantizer with step size a (which must be a positive odd integer) and initial level b (which must be an integer in the range $0, 1, \ldots, a-1$), we list all numbers in the form $an + b$ (where n is an integer) that are within the range $0, 1, \ldots, 100$, and then choose the number in the list which is the closest to x_0, and output that number as x_1.

a) What is the smallest positive odd integer a so that there are only 2 different possible values of x_1? You may choose the initial level b. Justify your assertion.

> $a = 35$, $b = 34$.
>
> If there are only two possible values: b and $a + b$, this means $2a + b > 100$. Since $b \leq a - 1$, we have $3a - 1 > 100$. Since a is odd, we have $a \geq 35$. We can check that $a = 35$, $b = 34$ gives only two possible values.

b) Compressing an image multiple times may degrade the quality of the image. Suppose now we apply a quantizer on x_0 (in the range $0, 1, \ldots, 100$) to get x_1, and then apply another quantizer on x_1 to get x_2, and so on. After we quantize k times, we get x_k. Your goal is to design a sequence of quantizers that is as bad as possible. The worst image compression software would compress any input image into an image that contains only a single colour. Therefore, you want the number of different possible values of x_k to be only 1.
For example, consider the sequence of quantizers (where $k = 3$):

1. Step size 39, initial level 30,
2. Step size 35, initial level 18,
3. Step size 51, initial level 40.

We can check that regardless of the value of x_0 (in the range $0, 1, \ldots, 100$), we must have $x_3 = 40$, and hence the goal is satisfied.

i) Consider the case where each of these quantizers must have the same step size a (you can choose a), and each of these quantizers may have a different initial level that you can choose. Fix $k = 2$ (you can quantize twice). What is the smallest positive odd integer a such that you can design a sequence of quantizers with only one possible value of x_k? Justify your assertion.

$a = 41$:

1. Step size 41, initial level 40, possible values: $40, 81$,
2. Step size 41, initial level 19, possible value: 60.

Let the initial levels be b_i, $i = 1, 2$. Note that 0 is quantized to b_1 after the first quantization, and to either b_2 (the smallest level) or $a + b_2$ (the second smallest level) after the second quantization. Note that $b_2 \leq a - 1$. If it is quantized to $a + b_2$ after the second quantization, then b_1 is closer to $a + b_2$ than to b_2, or equivalently, $b_1 \geq (a + 1)/2 + b_2$, which gives $a + b_2 = a/2 + (a + 1)/2 + b_2 - 1/2 \leq a/2 + b_1 - 1/2 \leq (3/2)(a - 1)$. In both cases, we know that 0 is quantized to a number $\leq (3/2)(a - 1)$ after two quantizations. It increases by at most $(3/2)(a - 1)$. By symmetry, we know that 100 decreases by at most $(3/2)(a - 1)$ after two quantizations.

Note that it is impossible to simultaneously have 0 quantized to the second smallest level ($a + b_2$) after the second quantization, and have 100 quantized to the second largest level after the second quantization. To see this, note that if we modify the quantization function in the second quantization so that a number is quantized to the closest number in the form $an + b$ where n is any (positive/negative/zero) integer, then the effect of second quantization will merely be increasing/decreasing the number by a fixed amount (we call the second quantization *increasing* or *decreasing* accordingly). More precisely, the second quantization is *increasing* if the closest number to b_1 among $an + b_2$ (for integer n) is larger than b_1. It is *decreasing* if the closest number to b_1 among $an + b_2$ (for integer n) is smaller than b_1. If 0 quantized to the second smallest level after the second quantization, then the second quantization must be increasing (note that it is possible for a decreasing quantization to increase a number if the closest number to b_1 among $an + b_2$ is $-a + b_2$, though this would mean that b_1 is quantized to b_2 which is the smallest level). However, if 100 quantized to the second largest level after the second quantization, then the second quantization must be decreasing, leading to a contradiction.

Therefore, after the second quantization, if 0 is quantized to the second smallest level (increased by at most $(3/2)(a - 1)$), then 100 is quantized to the largest level (decreased by at most $a - 1$). Since there is only one possible value, we must have

$$(3/2)(a - 1) + a - 1 \geq 100,$$

$$a \geq 41.$$

ii) Suppose now each quantizer must have a fixed step size $a = 11$, and you may only choose the initial levels of the quantizers (you may choose a different initial level each time). We allow k to be any positive integer. What is the smallest k such that you can design a sequence of quantizers with only one possible value of x_k? Justify your assertion.

After the first quantization, there are either 9 or 10 possibilities. If there are currently $m \leq 9$ possibilities, then after two quantizations, the number of possibilities is at least $m - 1$.[a] Therefore the smallest k is $8 \times 2 + 1 = 17$. For example, here is one sequence of possible values of x_i:

$$\text{Quantization } 1 : 10, 21, \ldots, 98$$
$$\text{Quantization } 2 : 15, 26, \ldots, 92$$
$$\text{Quantization } 3 : 20, 31, \ldots, 97$$
$$\vdots$$
$$\text{Quantization } 17 : 90.$$

[a] Each quantization (other than the first) can at most reduce the number of possibilities by 1. If there are currently ≤ 9 possibilities, then two consecutive quantizations can at most reduce the number of possibilities by 1. To see this, a decreasing quantization (see part b.i) can reduce the number of possibilities if the previous smallest possible number is ≤ 4, and if it reduces the number of possibilities, then the smallest possible number after the quantization is ≥ 6, implying that it is impossible for two decreasing quantization to both reduce the number of possibilities. It is also straightforward to check that it is impossible for a decreasing quantization and an increasing quantization to both reduce the number of possibilities if there are currently ≤ 9 possibilities.

iii) Suppose now each quantizer must have a fixed initial level $b = 0$. You first choose a positive odd integer c (the "maximum step size"). Then you can design a sequence of quantizers, where each quantizer has an initial level 0, and a step size at most c (you may choose a different step size each time as long as it is less than or equal to c). We allow k to be any positive integer. What is the smallest value of c such that you can design a sequence of quantizers with only one possible value of x_k? Justify your assertion.

Observe that there is only one possible value of x_k if and only if the sequence of quantizers maps $x_0 = 100$ to $x_k = 0$. We can use the following "greedy" strategy. Consider the largest possible value of x_i (let it be v). Initially, $v = 100$. Each time, we choose the quantizer with the smallest step size a that maps v to a number smaller than v. We repeat this until $v = 0$, and report the largest step size used. One way to speed up this strategy is to note that if we have already used a quantizer with step size $\tilde{a}$ previously, then we should not try $a = 1, 3, 5, \ldots$ in this order, but we should try $a = \tilde{a}, \tilde{a} - 2, \ldots, 1, \tilde{a} + 2, \tilde{a} + 4, \ldots$ since choosing $a = \tilde{a}$ is not going to increase the largest step size used. The answer is 15. The greedy strategy gives:

Step size = 3, largest possible value = 99

Step size = 7, largest possible value = 98

Step size = 13, largest possible value = 91

Step size = 11, largest possible value = 88

Step size = 3, largest possible value = 87

Step size = 7, largest possible value = 84

Step size = 13, largest possible value = 78

Step size = 11, largest possible value = 77

Step size = 5, largest possible value = 75

Step size = 9, largest possible value = 72

Step size = 7, largest possible value = 70

Step size = 13, largest possible value = 65

Step size = 9, largest possible value = 63

Step size = 15, largest possible value = 60

Step size = 11, largest possible value = 55

Step size = 13, largest possible value = 52

Step size = 15, largest possible value = 45

Step size = 13, largest possible value = 39

Step size = 9, largest possible value = 36

Step size = 15, largest possible value = 30

Step size = 13, largest possible value = 26

Step size = 11, largest possible value = 22

Step size = 15, largest possible value = 15

Step size = 13, largest possible value = 13

Step size = 11, largest possible value = 11

Step size = 9, largest possible value = 9

Step size = 7, largest possible value = 7

Step size = 15, largest possible value = 0

Question 2: Unbiased Coin

a) Imagine you possess a biased coin with a 1/3 probability of landing on heads and a 2/3 probability of landing on tails. You wish to participate in a game that requires an unbiased fair coin, but the only coin you have is the aforementioned biased one. Suppose you can throw the biased coin only two times. You would like to *simulate* a fair coin, but this may not be possible. What is the "fairest" coin you can simulate? In other words, after throwing the biased coin twice, you need to "declare" H or T, and you would like the probability of H and T to be as close as possible to each other. What strategy can you employ to achieve this?

With two coin throws, we get four possible outcomes with the following probabilities:

Outcomes	Probability
HH	1/9
HT	2/9
TH	2/9
TT	4/9

By direct inspection, we see that the best strategy is to declare Heads if the outcomes belong to $\{HH, HT, TH\}$ and Tails if the outcome is TT, simulating a coin with Tails probability of $\frac{4}{9} = \frac{1}{2} - \frac{1}{18}$.

b) Repeat the previous part if you are allowed to throw the biased coin only three times.

With three coin throws, we get four possible outcomes with the following probabilities:

Outcomes	Probability
HHH	1/27
HHT	2/27
HTH	2/27
HTT	4/27
THH	2/27
THT	4/27
TTH	4/27
TTT	8/27

By direct inspection, we see that the best strategy is to declare Heads if the outcomes belong to $\{HHH, HHT, HTH, TTT\}$ and Tails otherwise, simulating a coin with Heads probability of $\frac{13}{27} = \frac{1}{2} - \frac{1}{54}$.

c) Now, you have the freedom to toss the coin as many times as you desire. Can you devise a strategy that enables you to simulate a fair (unbiased) coin using the biased coin? Here's a hint: throw the biased coin twice. If both throws yield the same result (either two heads or two tails), disregard those tosses and repeat the procedure.

If you are allowed to throw the biased coin as many times as you like, you can use the following strategy to perfectly simulate a fair coin:

- Throw the biased coin twice.
- If the outcomes of the two throws are the same (either two Heads or two Tails), discard both throws and repeat the process.

- If the outcomes of the two throws are different (one Head and one Tail), declare the outcome of the first throw as the result of the fair coin.

By repeating this process, you ensure simulating a fair coin.

d) Imagine a scenario where you have two coins: one is fair and unbiased, while the other is biased. You don't have any prior knowledge about which coin is which, except that the biased coin has a 1/3 probability of landing on heads, while the unbiased coin has a 1/2 probability. Your goal is to determine which coin is the unbiased one. To make your guess, you randomly select one of the coins, toss it, and observe the result. Based on this outcome, what would be your most reasonable guess of which coin is the unbiased one?

If the outcome of the coin toss is heads, we conclude that it is the unbiased coin because the probability of obtaining heads with the biased coin is only 1/3, which is lower than the probability of heads with a fair coin (1/2). Conversely, if the outcome is tails, it is logical to conclude that it is the biased coin.

e) If you have the option to toss both coins (each once) in order to solve the previous question, what is the best guess (based on the outcomes) to maximize the probability of correctly identifying the unbiased coin?

If the results of the coin tosses are identical (either both heads or both tails), we gain no useful information from the outcomes and can only make a random guess regarding which coin is unbiased. However, if the outcomes are (heads ,tails), we can deduce that the first coin is the unbiased one. This conclusion stems from the fact that the probability of obtaining such outcomes in this scenario is higher, specifically calculated as the product of 1/2 (probability of heads on the unbiased coin) and 2/3 (probability of tails on the biased coin). This probability is greater than the likelihood of getting the same outcomes if the first coin is biased, calculated as the product of 1/3 (probability of heads on the biased coin) and 1/2 (probability of tails on the unbiased coin). Conversely, if the outcome is tails followed by heads, it is reasonable to conclude that the first coin is the biased coin.

Question 3: List Decoding

You have a ternary alphabet, say $\{0, 1, 2\}$. Consider

$$\{0, 1, 2\}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \{0, 1, 2\}, i = 1, 2, ..., n\}$$

be the set of sequences of length n from the ternary alphabet.

Adam has a collection of sequences $\mathcal{C}_n \subseteq \{0, 1, 2\}^n$ and wants to send a sequence $(x_1, x_2, \ldots, x_n)$ from $\mathcal{C}_n$ to his friend Eve. (Note that if a sequence is in $\mathcal{C}_n$, Adam can be asked to send that sequence; so $\mathcal{C}_n$ is not allowed to have unused sequences.)

However, during transmission, there is always noise, and if Adam sends i, for $i \in \{0, 1, 2\}$, at a certain position, Eve will receive one of the other two symbols at that position with probability $\frac{1}{2}$. (For example, if 1 was sent at some position, then Eve will receive either 0 or 2, each with probability $\frac{1}{2}$.) Assume that the noise at each position is independent of the noise at the other positions.

Below is an example: suppose $n = 2$, and Adam sends the sequence $(x_1, x_2) = (0, 1)$ to Eve. Then, Eve receives one of the following four sequences $(1, 0), (2, 0), (1, 2), (2, 2)$ with equal probability $1/4$. Similarly, if Adam sends the sequence $(x_1, x_2) = (0, 0)$ to Eve, she will receive one of the following four sequences $(1, 1), (2, 1), (1, 2), (2, 2)$ with an equal probability of $1/4$. Now, assume that $\mathcal{C}_2 = \{(0, 0), (0, 1)\}$ contains two possible sequences that Adam may send. Observe that if Eve receives $(2, 2)$, she will not be able to determine which sequence from $\mathcal{C}_2$ was transmitted. In this case, we say that an error occurs in the communication.

a) Eve wants to figure out exactly (with zero chance of error) which sequence was sent by Adam based on the sequence she receives. In this case, how large can the size of $\mathcal{C}_n$ be?

> Suppose that $\mathcal{C}_n$ contains two sequences $\mathbf{a} = (a_1, .., a_n)$ and $\mathbf{b} = (b_1, .., b_n)$. Now construct a sequence $\mathbf{s} = (s_1, ..., s_n)$, where s_i is different from a_i and b_i. When Adam sends the sequence $\mathbf{a}$ or $\mathbf{b}$, there is a non-zero probability (it is $\frac{1}{2^n}$ in this problem) for Eve to get the sequence $\mathbf{s}$. Then Even cannot precisely decode whether the sequence $\mathbf{a}$ or the sequence $\mathbf{b}$ was transmitted. Therefore $\mathcal{C}_n$ can only contain one sequence.

b) Disappointed by the above part, Eve is slightly less ambitious, and as long as she can narrow down the sequence sent by Adam to two possibilities, she is happy. **Prove** that in this case if $\mathcal{C}_n$ has the following property, then Eve can successfully complete her mission: For any three different sequences $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n)$, and $\mathbf{c} = (c_1, \ldots, c_n)$ in $\mathcal{C}_n$, there is some location $i, 1 \leq i \leq n$ such that (a_i, b_i, c_i) is a permutation of $(0, 1, 2)$ (that is, all the three values are distinct). We shall call such a collection of sequences, $\mathcal{C}_n$ to be *trifferent*.

> Upon receiving $\mathbf{s} = (s_1, .., s_n)$, Eve forms a list of all possible sequences $\mathbf{a} \in \mathcal{C}_n$ that could have resulted in the sequence $\mathbf{s}$ with non-zero probability. Suppose this list has three or more sequences in them: let the sequences be $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. Since $\mathcal{C}_n$ is trifferent, there is a location $i, 1 \leq i \leq n$ such that (a_i, b_i, c_i) is a permutation of $(0, 1, 2)$. Therefore one among $\{a_i, b_i, c_i\}$ will match s_i. W.l.o.g. let $c_i = s_i$. Then it is clear that the sequence $\mathbf{c} \in \mathcal{C}_n$ could not have resulted in the sequence $\mathbf{s}$. Therefore the lost can have at most two different sequences.

c) Prove the converse to the above statement, i.e. if Eve can successfully complete her mission, then the collection of sequences $\mathcal{C}_n$ is trifferent.

> Suppose the collection is not trifferent. Then there exists three (different) sequences $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ in $\mathcal{C}_n$, such that for every location $i, 1 \leq i \leq n$ the triple (a_i, b_i, c_i) is missing at least one element in $(0, 1, 2)$; denote the missing element (if there are multiple pick any) to be s_i. Now note that any of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ can result in sequence $\mathbf{s}$. Hence upon receiving $\mathbf{s}$, Even cannot eliminate with certainty any of the three sequences. This contradicts the fact that Eve can complete her mission.

d) Let $T(n)$, $n \geq 1$. be defined as the largest value of the size of $\mathcal{C}_n$, where $\mathcal{C}_n$ is a set of sequences of length n that is trifferent. Then show the following:

- $T(1) = 3$

Consider the set $\{0, 1, 2\}$. This has cardinality 3, and Eve can declare her list to be the complement of what she observes. Since there are at most 3 distinct sequences of length 1, $T(1) = 3$.

- $T(2) = 4$

Consider the set $\{00, 10, 21, 22\}$. Clearly, this collection works.

Suppose there is a set of size 5 that works. Note that all of 0 or 1 or 2 must occur at least once in first position. Otherwise, the second position will have to yield a proof of trifference, but $T(1) = 3 \leq 5$.

Let us divide into 2 cases:

Case 1: there is an element that occurs three times in the first position.

In this case our collection has to look like $\{00, 01, 02, 1x, 2y\}$. By noting that 2nd, 3rd, and 4th sequences to be trifferent, we infer that x must be zero. By noting that 2nd, 3rd, and 5th sequences to be trifferent, we infer that y must be zero. Therefore our list must be $\{00, 01, 02, 10, 20\}$. However now, the collection 1st, 2nd, and 4th is not trifferent.

Case 2: there are two elements that occurs two times each in the first position.

In this case our collection has to look like $\{0p, 0q, 1r, 1s, 2t\}$. But cosnidering the first 4 sequences, we are forced to have the collection $\{p, q, r, s\}$ to be trifferent. However $T(1) = 3 < 4$.

e) Show that, for any $n \geq 2$,

$$T(n) \leq \frac{3}{2}T(n-1).$$

Consider the last position. One of the symbols must occur as few or fewer than all others. W.l.o.g. let that symbol be 2. Let $\hat{\mathcal{C}}_n \subset \mathcal{C}_n$ be the collection whose last symbol is 0 or 1. Clearly $|\hat{\mathcal{C}}_n| \geq \frac{2}{3}|\mathcal{C}_n|$. Now, consider any three sequences in $\hat{\mathcal{C}}_n$. The proof of their trifference must come from the first $n-1$ positions. Let $\hat{\mathcal{C}}_{n-1}$ be formed from $\hat{\mathcal{C}}_n$ by removing the last symbol from each sequence. Clearly $\hat{\mathcal{C}}_{n-1}$ is a trifferent collection (and hence, contains no repeated sequences). Therefore

$$T(n-1) \geq |\hat{\mathcal{C}}_{n-1}| \geq \frac{2}{3}|\mathcal{C}_n|.$$

Since this holds for all trifference $\mathcal{C}_n$, we see that

$$T(n-1) \geq \frac{2}{3}T(n),$$

as required.

f) Determine the value of $T(4)$.

From the previous part, we get that $T(3) \leq \frac{3}{2}T(2) = 6$, and $T(4) \leq \frac{3}{2}T(3) = 9$.

The set of sequences $\{210, 120, 201, 021, 102, 012\}$ are trifferent.

The set of sequences $\{0012, 0120, 0201, 1210, 1102, 1021, 2000, 2111, 2222\}$ are trifferent.

Remark: This question is related to perfect hashing and the best bound known today comes from the concatenation of the elements of $T(4)$. One can show that asymptotically $T(n)$ grows at least as fast as $\left(\frac{9}{5}\right)^{\frac{n}{4}}$. The key point is that once Eve allows for even an uncertainty of two symbols, the communication rate drastically increases.

Question 4: Periodic functions

A function $f(x)$ is called periodic if there is a value $T > 0$, such that $f(x+T) = f(x)$ for every x. Any positive value T such that $f(x+T) = f(x)$ for every x, is called a *period* of the function.

a) Show that if 33 and 27 are periods of $f(x)$, then 3 is also a period of $f(x)$.

It is easy to see that $f(x) = f(x + 33m + 27n)$, for $m, n \in \mathbb{Z}$. This can be shown for $m, n \geq 0$ by induction. For the other cases of signs of m, n, by shifting the starting point. Therefore, as long as $33m + 27n$ is positive, it is a period. Taking $n = 5, m = -4$ we see that 3 is a period of $f(x)$.

b) Show that if $\sqrt{2}$ and 1 are periods of a continuous function $f(x)$, then $f(x)$ must be the constant function.

Using the same reasoning as in the previous part, any positive value of $m\sqrt{2} + n$, with $m, n \in \mathbb{Z}$, is a period of $f(x)$. Note that $(\sqrt{2} - 1)^k$, for $k \in \mathbb{N}$, is of the form $m\sqrt{2} + n$ (by the Binomial theorem) and hence is a period.

Suppose that we have a non-constant continuous function satisfying the above conditions. Therefore, there exists $b, a \in \mathbb{R}$, with $f(b) > f(a)$. Let $f(b) - f(a) = \epsilon$. By the continuity of $f(x)$, we know that there exists a $\delta > 0$, such that for all $\{x : |x - b| < \delta\}$, we have $|f(b) - f(x)| < \frac{\epsilon}{2}$. Choose k such that $T_k := (\sqrt{2} - 1)^k < \frac{\delta}{2}$. Now consider the collection of points $\{a + nT_k\}_{n \in \mathbb{Z}}$. These points are spaced T_k apart and, therefore, the maximum distance from b to one such point is at most T_k. Let n_0 be such that $|b - (a + n_0 T_k)| \leq T_k$. Since T_k is a period, we have $f(a + n_o T_k) = f(a)$. Therefore, $|f(b) - f(a + n_0 T_k)| = \epsilon$, while $|b - (a + n_0 T_k)| \leq T_k < \frac{\delta}{2}$. This contradicts the statement that for all $\{x : |x - b| < \delta\}$, we have $|f(b) - f(x)| < \frac{\epsilon}{2}$.

c) Construct a non-constant function (that need not be continuous) such that $\sqrt{2}$ and 1 are both its periods.

$$f(x) = \begin{cases} 1 & x = m\sqrt{2} + n, \text{ where } m, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} .$$

d) Prove that $f(x) = \sin(x) + \sin(\sqrt{2}\,x)$ is not periodic.

Solution 1: We have $f'(x) = \cos(x) + \sqrt{2}\cos(\sqrt{2}x) \leq 1 + \sqrt{2}$, and $f'(x) = 1 + \sqrt{2}$ if and only if $x = 2\pi n$ and $\sqrt{2}x = 2\pi m$ for some integers n, m, which gives $m/n = \sqrt{2}$ if $n \neq 0$. Since $\sqrt{2}$ is irrational, we must have $n = 0$, and $m = 0$. Hence, $f'(x) = 1 + \sqrt{2}$ if and only if $x = 0$. $f'(x)$ cannot be periodic. $f(x)$ cannot be periodic either.

Solution 2: Assume otherwise. Let $T > 0$ be a period of $f(x)$. Then we have, for all x,

$$\begin{aligned} f(x) &= 2\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right) \\ &= 2\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)(x+T)\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)(x+T)\right). \end{aligned} \tag{1}$$

Taking $x = 0$, we obtain that

$$0 = 2\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)T\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)T\right).$$

Case 1: $\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)T\right)=0$.

In this case, we have $T=\frac{2k\pi}{1+\sqrt{2}}$, for some $k\in\mathbb{N}$. Substituting this back into (1), we get

$$\begin{aligned}
&\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right)\\
&=\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x+k\pi\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x+k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\\
&=(-1)^k\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x+k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).
\end{aligned}$$

Therefore for all x not of the form $\frac{2\ell p}{\sqrt{2}+1},\ell\in\mathbb{Z}$, we have

$$\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right)=(-1)^k\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x+k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

Since the two functions on both sides are continuous, the equality must continue to hold for all x. Setting $x=0$, we obtain

$$1=(-1)^k\cos\left(k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}\right),\ \text{or equivaently}\ (-1)^k=\cos(k\pi)=\cos\left(k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

This is only true if

$$k\pi\frac{\sqrt{2}-1}{\sqrt{2}+1}=k\pi+2\ell\pi,$$

for some $\ell\in\mathbb{Z}$. Rewriting the above yields

$$\sqrt{2}=\frac{-k-\ell}{\ell},$$

a contradiction to the irrationality of $\sqrt{2}$.

Case 2: $\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)T\right)=0$.

The proof is similar to the first case.

In this case, we have $T=\frac{(2k-1)\pi}{\sqrt{2}-1}$, for some $k\in\mathbb{N}$. Substituting this back into (1), we get

$$\begin{aligned}
&\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right)\\
&=\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x+\frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{2k-1}{2}\pi\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x+\frac{(2k-1)\pi}{2}\right)\\
&=(-1)^k\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x+\frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{2k-1}{2}\pi\right)\sin\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x\right)\cos\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right)\\
&=(-1)^k\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)x+\frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{2k-1}{2}\pi\right)\sin\left(\left(\frac{\sqrt{2}-1}{2}\right)x\right).
\end{aligned}$$

Set $x = \frac{\pi}{\sqrt{2}-1}$. Then we get

$$\sin\left(\left(\frac{1+\sqrt{2}}{2}\right)\frac{\pi}{\sqrt{2}-1} + \frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{(2k-1)}{2}\pi\right) = 0.$$

This holds if and only if

$$\left(\frac{1+\sqrt{2}}{2}\right)\frac{\pi}{\sqrt{2}-1} + \frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{(2k-1)}{2}\pi = \ell\pi,$$

for some $\ell \in \mathbb{Z}$. Rewriting this yields

$$\sqrt{2} = \frac{k+\ell}{k-\ell},$$

a contradiction, as before.