The Competition on the Mathematics of Information 2025

The Chinese University of Hong Kong

1 March, 2025

Name: ______________________________

Contestant Number: ______________________________

School Name: ______________________________

Question	Points
1	70
2	70
3	70
4	70
Total	280

Instructions:

1. This examination booklet contains 5 pages, including this page. Please write your solutions on the answer sheets given to you. You should write the solution to each problem **only** on its designated answer sheet. If more answer sheets are required, please ask the invigilators. You can use this booklet as scratch paper. Please ask if you require more scratch paper.
2. You have **three (3) hours** to complete the examination. As a courtesy to your classmates, we ask that you not leave during the last fifteen minutes.
3. Calculators are **not** allowed in the exam.
4. Please clip your answer papers with the paper clip provided as well as the cover page and return to us at the end of the examination.
5. Please sign the below Honor Code statement.

In recognition of and in the spirit of the Honor Code, I certify that I will neither give nor receive unpermitted aid on this examination.

Signature: ______________________________

Question 1: Sums of powers

Let x, y, z be real numbers that satisfy

$$
\begin{aligned}
x + y + z &= a, \\
x^2 + y^2 + z^2 &= b^2, \\
x^3 + y^3 + z^3 &= c^3,
\end{aligned}
$$

for some real numbers a, b, c.

(a) Find the value of $x^4 + y^4 + z^4$ in terms of a, b, c.

(b) Let $S_n = x^n + y^n + z^n$, for integers $n \geq 4$. Show that

$$S_n = p_1(a, b, c)S_{n-1} + p_2(a, b, c)S_{n-2} + p_3(a, b, c)S_{n-3},$$

for some (multivariate) polynomials $p_1(a, b, c), p_2(a, b, c), p_3(a, b, c)$. Furthermore, determine the three polynomials.

Question 2: Guess the number

In a number guessing game, your task is to correctly guess an integer x in the range $3, 4, \dots, n$, where $n \geq 4$ is a given integer (you initially know n, but not x). You are allowed to ask two kinds of questions:

- A **threshold question** is in the form "is $x \geq a$?". You are allowed to choose any integer a when you ask such a question. You will receive an answer "yes" if $x \geq a$, or "no" if $x < a$.
- The **smallest prime factor question** is the question "what is the smallest prime factor of x?". Let $f(x)$ be the smallest prime factor of x. You will receive the value of $f(x)$ as the answer.

Your goal is to guess x correctly after asking the fewest questions. An m-**question strategy** is a strategy that guarantees a correct guess after asking at most m questions. You are allowed to choose which question to ask next based on the answers to your previous questions.

For example, when $n = 6$, one strategy is to first ask the smallest prime factor question. If the answer is 3, you guess $x = 3$. If the answer is 5, you guess $x = 5$. If the answer is 2, you further ask whether $x \geq 6$. If yes, you guess $x = 6$. If no, guess $x = 4$. This is a 2-question strategy since you can guarantee a correct guess after at most $m = 2$ questions.

a) For $n = 11$, what is the minimum number of questions needed (i.e., what is the smallest m such that there exists an m-question strategy)? Justify your assertion. (To justify your assertion, you should explain your strategy, and argue that there is no strategy that requires fewer questions.)

b) Now consider general n, and you must ask the smallest prime factor question first and exactly once (i.e., your strategy must begin with the smallest prime factor question). Find the minimum number of questions needed, in terms of n. Justify your assertion.
Hint: You may express your answer in terms of the floor function $\lfloor t \rfloor$ (which is the greatest integer that is not greater than t), the ceiling function $\lceil t \rceil$ (which is the smallest integer that is not smaller than t), and logarithms. Note that $\lceil \log_2 t \rceil$ is the smallest integer a such that $t \leq 2^a$.

c) Now consider general n, and you must ask the smallest prime factor question **last** and exactly once (i.e., you must ask the smallest prime factor question, and you must guess x right after the smallest prime factor question). Find the minimum number of questions needed, in terms of n. Justify your assertion.

d) Now consider general n, and suppose you cannot ask the smallest prime factor question. Your goal now is not to guess x, but to guess $f(x)$ (the smallest prime factor of x) within a distance 1, that is, after the questions, you say an integer q, and you are considered correct if $|f(x) - q| \leq 1$ (i.e., the absolute difference between $f(x)$ and q is at most 1). Find the minimum number of questions needed, in terms of n. Justify your assertion.

Question 3: Sequences with maximum pairwise distance

We say that $(a_1, a_2, \cdots, a_n)$ is a binary sequence of length n if each entry a_i is either 0 or 1. Given two binary sequences $\vec{a} = (a_1, a_2, \cdots, a_n)$ and $\vec{b} = (b_1, b_2, \cdots, b_n)$, their distance is equal to the number of positions in which they differ, i.e., the number of $1 \leq i \leq n$ where $a_i \neq b_i$. In other words,

$$d(\vec{a}, \vec{b}) = \sum_i |a_i - b_i|.$$

For instance, the distance between $(0, 0, 0, \cdots, 0)$ and $(1, 1, 1, \cdots, 1)$ is n because the two sequences differ in every index. The distance between binary sequences $\vec{a} = (a_1, a_2, \cdots, a_n)$ and $\vec{b} = (b_1, b_2, \cdots, b_n)$ is denoted by $d(\vec{a}, \vec{b})$. Note that $d(\vec{a}, \vec{b})$ is an integer between 0 and n.

a) Let n be an integer divisible by 3. Find three sequences $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ of length n such that the minimum pairwise distance between them is maximized, i.e., find $\vec{a}_j = (a_{j1}, a_{j2}, \cdots, a_{jn})$ for $j = 1, 2, 3$ such that

$$\min_{t \neq k} d(\vec{a}_t, \vec{a}_k) = \min\left(d(\vec{a}_1, \vec{a}_2), d(\vec{a}_1, \vec{a}_3), d(\vec{a}_2, \vec{a}_3)\right)$$

is maximized.

b) Assume that we have r sequences $\vec{a}_i$ for $i = 1, 2, 3, \cdots, r$ such that the minimum pairwise distance between them is at least c, i.e.,

$$\min_{u \neq w} d(\vec{a}_u, \vec{a}_w) \geq c.$$

Show that

$$r \left(\sum_{j=0}^{t} \binom{n}{j} \right) \leq 2^n$$

where $t = \left\lfloor \frac{c-1}{2} \right\rfloor$ and $\binom{n}{j} = n!/(j!(n-j)!)$ is the binomial coefficient.

Hint: For any sequence $\vec{a}_u$, $\sum_{j=0}^{t} \binom{n}{j}$ is the number of binary sequences whose distance from $\vec{a}_u$ is at most t. This counts all possible ways of selecting up to t positions out of n positions to flip (change from 0 to 1, or 1 to 0).

c) Let n be an integer divisible by 5. Find three sequences $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ of length n such that

$$\min\left(2d(\vec{a}_1, \vec{a}_2), d(\vec{a}_1, \vec{a}_3), d(\vec{a}_2, \vec{a}_3)\right)$$

is maximized. Note the factor 2 in front of $d(\vec{a}_1, \vec{a}_2)$.

d) Assume that we have r sequences $\vec{a}_i$ for $i = 1, 2, 3, \cdots, r$ such that the pairwise distance between any two of them is exactly k. Moreover, each sequence $\vec{a}_i$ has exactly h ones and $n - h$ zeros. Assuming that $2h^2 - 2nh + nk \geq 0$, show that

$$r \leq \frac{nk}{2h^2 - 2nh + nk}.$$

Question 4: Functional Equation

Find all functions $g : [0, 1] \mapsto \mathbb{R}$ that satisfy all the following conditions:

(a) g is continuous.

(b) $g(0) = g(1) = 0$.

(c) $g(x) = g\left(\frac{x}{2}\right) + \left(1 - \frac{x}{2}\right) g\left(\frac{x}{2-x}\right)$, for all $x \in [0, 1]$.

(d) $g(x) = g(1 - x)$ for all $x \in [0, 1]$.