Power Control for non-Gaussian Interference

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***Abstract*—This paper investigates the power control problem involving a small number of active users whereby the standard Gaussian interference noise assumption does not hold. The model also allows for different user transmission rates. We analyze the situation under which the system can be asymptotically error-free. Subsequently, we formulate a power control optimization problem and propose an Iterative Descent Algorithm for solution. We prove that under suitable conditions, the power control optimization problem has a unique solution which is achieved by the proposed algorithm. Simulations are carried out and compared with the power control results under the classical Gaussian assumption.**

I. Introduction

In the literature of the power control problem in wireless communication systems (see for example the references in [1]–[3]), mutual user interference is typically modeled as a Gaussian process so that for any given user, the combined interference from the other users is treated as an additive Gaussian noise having a uniform power spectral density over the frequency band of interest. This Gaussian model can be partially justified by the Central Limit Theorem if the interference comes from mutually independent, identically distributed user signal processes and if the number of such users is large. However, in practice the number of interfering users may be small or they may not be independent identically distributed. The Gaussian model is not accurate enough when the Central Limit Theorem is not applicable [4].

This observation highlights the need for the study of a system model where the interference is non-Gaussian. We derive an exact expression for the average bit error probability (BEP) for such a system. A related power control optimization problem based on this new model is considered. A major finding of our work is that the non-Gaussian model has a significantly different performance characteristics from the traditional classical model. This is reflected in two major aspects. The first concerns the feasible BEP region. Under the Gaussian model, the BEP is a monotonically decreasing function of the signal-to-interference-plus-noise power ratio (SINR), and it is a well-known result of power balancing that the maximum of the minimal SINR is related to the dominant eigenvalue of the channel gain matrix and therefore is bounded. Hence, it is not possible to make the BEP's arbitrarily small uniformly. While for the non-Gaussian model, it is possible that the BEP's of all users go simultaneously to zero under certain technical condition which is explicitly characterized here.

The second performance difference shows up when the users have different transmission bit rates. In the Gaussian model, the SINR is defined to be the power of the message signal divided by the sum of the powers of interfering users plus the thermal noise. Consider the example where the thermal noise are negligible. For a given power setting, one can obtain a corresponding SINR vector. If we change the bit rate of one of the users but keep its power level fixed, then it follows from the defining formula that the SINR's of other users will not change and so will their BEP's. Simulation results for the non-Gaussian model however show otherwise. That is, varying transmission bit rate does change the BEP performance of a system. On the other hand, in practice, when an interfering user transmits at a bit rate larger than that of the intended user, its signal is spread over a larger bandwidth. Hence, part of its power will fall out of the bandwidth of the receiver of interest and thus be filtered. However, in the Gaussian model, the whole power of the interfering user is taken into account, which ignores the bandwidth effect. This issue is considered in the non-Gaussian model. One goal of this paper is to investigate the effect of multirate on system performance.

The power control problem presented in this paper is the basic optimization problem that aims to minimize the total power of all users subject to the BEP's satisfying given bounds. We prove that when the channel gains satisfy certain conditions, there is a unique optimal solution. An algorithm, entitled the Iterative Descent Algorithm, is proposed and is shown to be convergent to the optimal solution.

The rest of the paper is organized as follows. The system model and the error probability calculation are presented in Section II. Section III describes the power control problem. Analysis of the optimal solution is also presented. Simulation results are provided in Section IV. We compare the results between the Gaussian model and the non-Gaussian model. Finally, in Section V, we give some concluding remarks.

II. System Model and Error Probability Calculation

A. System Model

Consider a wireless communication system with n transmitters $\{\mathrm{tran}_i, i = 1, \cdots, n\}$ and n receivers $\{\mathrm{rec}_i, i = 1, \cdots, n\}$, in which, tran_i communicates to rec_i and all the transmissions share the same wireless radio spectrum. We refer user i to be the pair $(\mathrm{tran}_i, \mathrm{rec}_i)$. Let x_i^2 be the transmitted power of tran_i where $x_i \geq 0$ is the amplitude of the transmitted signal. Let R_i be the transmission bit rate of

tran_i. Assume slow and flat fading. Let g_{ij}^2 be the channel gain between tran_j and rec_i where $g_{ij} \geq 0$ is the attenuation factor on x_j. We consider a snapshot of the system, and thus g_{ij} is treated as a constant. Its magnitude reflects the effect of path loss, shadow fading and antenna gains.

All transmitters apply binary phase-shift keying (BPSK) modulation. Let p_{T_i} be the unit-amplitude rectangular pulse of duration $T_i = 1/R_i$ and $\{b_i^k\}_{k=0}^{\infty}$ be the information sequence of tran_i, where b_i^k is uniformly distributed on $\{\pm 1\}$. Assume there is no frequency offset and phase offset in all the transmitters and receivers. Thus the carrier is suppressed for notational economy. The baseband signal $s_i(t)$ of tran_i is

$$s_i(t) = x_i a_i(t) \tag{1}$$

where

$$a_i(t) = \sum_{k=0}^{\infty} b_i^k p_{T_i}(t - kT_i). \tag{2}$$

The transmitted signals from all the transmitters are not necessarily synchronized. At rec_i, the received baseband signal $r_i(t)$ is

$$r_i(t) = \sum_{j=1}^{n} g_{ij} x_j a_j(t - \tau_{ij}) + n_i(t), \tag{3}$$

where τ_{ij} is the time delay of $s_j(t)$ at rec_i, and $n_i(t)$ is the additive white Gaussian noise (AWGN) with two-sided power spectral density $N_0/2$.

A receiver demodulates the received baseband signal using a matched filter, followed by a threshold decision. The impulse response $s_i^0(t)$ of the filter of rec_i is a rectangular pulse of amplitude 1 and duration T_i. Without loss of generality, we assume $\tau_{ii} = 0$, i.e., the matched filter of rec_i is synchronized to the arrival signal transmitted by tran_i, and assume τ_{ij} $(j \neq i)$ is uniformly distributed in $[0, T_j]$. For the analysis, we consider a bit interval as $[0, T_i]$ for convenience. Assume $b_i = 1$, the input to the decision device for rec_i is

$$y_i = \int_0^{T_i} r_i(t) s_i^0(t) \mathrm{d}t = W_{ii} + \sum_{j \neq i} W_{ij} + Z_i \tag{4}$$

where

$$W_{ij} = g_{ij} x_j \int_0^{T_i} a_j(t - \tau_{ij}) \mathrm{d}t, \tag{5}$$

$$Z_i = \int_0^{T_i} n_i(t) s_i^0(t) \mathrm{d}t. \tag{6}$$

Since $\tau_{ii} = 0$, $W_{ii} = T_i g_{ii} x_i$. Unlike classical models, the interference term $\sum_{j \neq i} W_{ij}$ is not assumed to be a Gaussian random variable. For all $j \neq i$, W_{ij} is a random variable depends on τ_{ij}, T_j and the information bits of tran_j. Several typical cases of the integral $\int_0^{T_i} a_j(t - \tau_{ij}) \mathrm{d}t$ are illustrated in Fig. 1.

For the threshold decision, an error occurs if $y_i < 0$ when $b_i = 1$, or if $y_i > 0$ when $b_i = -1$. Since b_i takes ± 1 with equal probability, the average BEP is equal to the probability of having $y_i < 0$ when $b_i = 1$. Since Z_i is a Gaussian random variable with zero mean and variance $N_0 T_i / 2$, the BEP of user i conditioned on $\sum_{j \neq i} W_{ij}$ is

$$\Pr\left(y_i < 0 | \sum_{j \neq i} W_{ij}, b_i = 1\right) = Q\left(\frac{\sum_j W_{ij}}{\sqrt{\frac{N_0 T_i}{2}}}\right), \tag{7}$$

where $Q(\cdot)$ is the complementary error function defined as $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} \mathrm{d}u$.

Hence, the average BEP is

$$\lambda_i = \mathrm{E}[\Pr(y_i < 0 | \sum_{j \neq i} W_{ij}, b_i = 1)], \tag{8}$$

where the expectation is over $\{b_j : j \neq i\}$ and $\{\tau_{ij} : j \neq i\}$.

B. Asymptotically Error-Free System

Let $\mathbf{x} = (x_1, \ldots, x_n)$. Define a function $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with component functions $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$, where λ_i is defined in (8). Scalar operators, such as " $>$", " $\geq$" or " $=$" are applied to vectors component-wise. We use "$|\cdot|$" to denote the absolute value and "$||\cdot||$" to denote the Euclidean norm.

Definition 1. *An n-user wireless communication system is said to be asymptotically error-free if*

$$\inf_{\mathbf{x} > 0} \max_{i=1,\cdots,n} \lambda_i = 0. \tag{9}$$

Remark: This definition of asymptotically error-free transmission is focused on the decision error at the physical layer. It is different from the information-theoretic concept of source-symbol error probability (SEP) in coding theory.

A matrix $A = (a_{ij})$ is said to be row diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for all } i. \tag{10}$$

A square matrix is called a Z-matrix if all off-diagonal entries are less than or equal to zero. A Z-matrix A is called an M-matrix if it satisfies any one of the following equivalent conditions:

1) The eigenvalues of A all have positive real parts.
2) $A\mathbf{v} \geq 0$ implies $\mathbf{v} \geq 0$.
3) There exists a vector $\mathbf{v}$ with positive entries such that $A\mathbf{v} > 0$.
4) The diagonal entries of A are positive and AD is row diagonally dominant for some positive diagonal matrix D.

For more discussion of M-matrix, refer to [5].

Definition 2. *The* character matrix *of an n-user wireless communication system with channel gain matrix $G = (g_{ij})$ is an $n \times n$ matrix $C = (c_{ij})$ with $c_{ii} = g_{ii}$ for all i and $c_{ij} = -g_{ij}$ for $j \neq i$.*

Obviously, C is a Z-matrix.

Theorem 1. *An n-user wireless communication system is asymptotically error-free if and only if its character matrix is an M-matrix.*

Proof: Consider an n-user wireless communication system with channel gain matrix $G = (g_{ij})$ and character matrix C. First fix a receiver i. We see that for $j \neq i$,

$$W_{ij} \geq -T_i g_{ij} x_j, \tag{11}$$

where the equality is obtained when $a_j(t - \tau_{ij}) = -1$ for $0 \leq t \leq T_i$, that is, $b_j^k = -1$ for $k < T_i/T_j + 1$.

Since $Q(\cdot)$ is a decreasing function,

$$\lambda_i = \mathrm{E}\left[Q\left(\sum_j W_{ij}/\sqrt{N_0 T_i/2}\right)\right] \tag{12}$$

$$< E\left[Q\left(T_i(g_{ii}x_i - \sum_{j\neq i} g_{ij}x_j)/\sqrt{N_0 T_i/2}\right)\right] \tag{13}$$

$$= Q\left(\tilde{y}_i\sqrt{2T_i/N_0}\right), \tag{14}$$

where $\tilde{y}_i = g_{ii}x_i - \sum_{j\neq i} g_{ij}x_j$.

Let $\tilde{\mathbf{y}} = [\tilde{y}_1, \cdots, \tilde{y}_n]^\top$. It follows from Definition 2 that $\tilde{\mathbf{y}} = C\mathbf{x}$.

By the third equivalent definition of M-matrix, if C is an M-matrix, there exists $\mathbf{x}^* > 0$ such that $\tilde{\mathbf{y}}^* = C\mathbf{x}^* > 0$. Let $\{m\mathbf{x}^* : m > 0\}$ be a sequence of input power with $m \to \infty$. Then, $\lambda_i < Q(m\tilde{y}_i^*\sqrt{2T_i/N_0}) \to 0$ as $m \to \infty$ for all i. That is, the system is asymptotically error-free.

On the other hand, if C is not an M-matrix, for any $\mathbf{x} > 0$, there exists at least one entry in $C\mathbf{x}$, say $\tilde{y}_i$, which is not positive. Let

$$P_i = \Pr\{b_j^k = -1 : j \neq i, (k-1)T_j < T_i\} \tag{15}$$

$$\geq 2^{-\sum_{i\neq j}(\lceil T_i/T_j\rceil + 1)} \tag{16}$$

$$= P_i'. \tag{17}$$

Then,

$$\lambda_i > P_i Q\left(\tilde{y}_i\sqrt{2T_i/N_0}\right) \tag{18}$$

$$> P_i Q(0) \tag{19}$$

$$> P_i/2. \tag{20}$$

This means that the system is not asymptotically error-free. ∎

Remark: In the Gaussian model, the BEP is a monotonically decreasing function of SINR, and it is a well-known result of power balancing that the maximum of the minimal SINR is related to the dominant eigenvalue of the channel gain matrix and therefore is bounded. Hence, no matter what the power setting is, the BEP's under the Gaussian model fail to approach zero uniformly. However, in the non-Gaussian model, the asymptotically error-free property can be achieved.

III. Power Control Problem

In the subsequent discussion, we consider systems with character matrix C being an M-matrix.

A. Problem Statement

There are different formulations for the power control problem. In this paper, our goal is to minimize the total transmitted power of a system while maintaining an acceptable quality-of-service (QoS) for each user.

Let ϵ be the target BEP. The power control problem can be stated as

$$\begin{aligned} \min \quad & \sum_i x_i^2 \\ \text{s.t.} \quad & \lambda_i(\mathbf{x}) \leq \epsilon \qquad i = 1, \ldots, n. \end{aligned} \tag{21}$$

In general, the function λ_i's are non-convex on $\{\mathbf{x} : \mathbf{x} > 0\}$, and thus (21) is not a convex optimization problem. However, note that the Q-function is convex on $\mathbb{R}^+$. We can add a constraint such that the new problem is convex.

Define

$$\mathcal{X}_C = \{\mathbf{x} : C\mathbf{x} > 0\}. \tag{22}$$

Since C is an M-matrix, $\mathcal{X}_C$ is non-empty. Furthermore, $\mathcal{X}_C$ is an intersection of n halfspaces and hence convex.

Lemma 1. *For any user i, over $\mathbf{x} \in \mathcal{X}_C$, λ_i is a strictly and monotonically decreasing function of x_i and a strictly and monotonically increasing function of x_k for $k \neq i$.*

Proof: See Appendix A. ∎

Lemma 2. *For any user i, λ_i is convex on $\mathcal{X}_C$.*

Proof: Fix a user i. Let $q_{ij} = g_{ij}\int_0^{T_i} a_j(t - \tau_{ij})\mathrm{d}t$ and $\mathbf{q}_i = [q_{i1}, \ldots, q_{in}]$. Given $\mathbf{q}_i$, by (5), $\sum_j W_{ij} = \mathbf{q}_i\mathbf{x}$ is a linear function of $\mathbf{x}$, and thus the range of $\sum_j W_{ij}$ over $\mathcal{X}_C$ is convex. Moreover, for any $\mathbf{x} \in \mathcal{X}_C$, we have,

$$\sum_j W_{ij} = T_i g_{ii} x_i + \sum_{j\neq i} W_{ij} \tag{23}$$

$$\geq T_i g_{ii} x_i - T_i \sum_{j\neq i} g_{ij} x_j \tag{24}$$

$$> 0. \tag{25}$$

Since $Q(x)$ is convex on $x > 0$, $Q(\sum_j W_{ij}/\sqrt{N_0 T_i/2})$ is convex on $\mathcal{X}_C$. Therefore, by the definition of λ_i in (8), it is ready to verify that the expected BEP function λ_i is convex on $\mathcal{X}_C$. ∎

Consequently, the following problem is a convex optimization problem,

$$\begin{aligned} \min \quad & \sum_i x_i^2 \\ \text{s.t.} \quad & \lambda_i(\mathbf{x}) \leq \epsilon \qquad i = 1, \ldots, n \\ & \mathbf{x} \in \mathcal{X}_C. \end{aligned} \tag{26}$$

Let $\mathcal{X}_\epsilon = \{\mathbf{x} : \lambda_i(\mathbf{x}) \leq \epsilon, i = 1, \ldots, n\}$. Since C is an M-matrix, by Theorem 1, $\mathcal{X}_\epsilon$ is non-empty

Proposition 1. *There exists ϵ_0 such that for $\epsilon \leq \epsilon_0$, $\mathcal{X}_\epsilon \subset \mathcal{X}_C$, i.e., (21) and (26) have the same optimal solution.*

Proof: Let $\epsilon \leq \epsilon_0 \triangleq \min_i(P_i'/2)$, where P_i' is defined in (17). If there exists an $\mathbf{x} \in \mathcal{X}_\epsilon$ but $\mathbf{x} \notin \mathcal{X}_C$, at least one entry in $C\mathbf{x}$, say the ith entry, is not positive. By (16) and (20), $\lambda_i(\mathbf{x}) > P_i'/2 \geq \epsilon$, i.e., a contradiction to $\mathbf{x} \in \mathcal{X}_\epsilon$. Hence $\mathcal{X}_\epsilon \subset \mathcal{X}_C$. ∎

Remark: Note from (17), ϵ_0 as defined above concerns the rate ratio between users. In practice, the required BEP is of the order at least 10^{-3}. Since we are discussing the system of a small number of users, it is reasonable to assume $\epsilon \leq \epsilon_0$. Hence in the subsequent discussion, (21) and (26) are equivalent.

For the convex optimization problem with strictly convex objective function, it is known that the optimal solution is unique [6]. Hence, if the optimal solution for (26) is attained, it is unique.

B. Properties of the Optimal Solution

In this section, we want to prove that the optimal solution for (26) is attained and satisfies the inequality constraint with equality, i.e., $\lambda_i(\mathbf{x}) = \epsilon$ for $i = 1, \ldots, n$. First, we propose the **Iterative Descent Algorithm:**

Input: $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon$.

1) Set $k \leftarrow 1$,
2) $i = |k|_{mod\ n}$.
   If $\lambda_i^{(k)} < \epsilon$,
   let $\mathbf{x}^{(k+1)} = [x_1^{(k)}, \ldots, x_{i-1}^{(k)}, x_i^{(k+1)}, x_{i+1}^{(k)}, \ldots, x_n^{(k)}]$ s.t. $\lambda_i^{(k+1)} = \epsilon$;
   otherwise $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)}$.
3) $k \leftarrow k + 1$.

Lemma 3. *For each user i, the Iterative Descent Algorithm generates a sequence $\{x_i^{(k)}\}_k$. When $\epsilon \leq \epsilon_0$, the sequence $\{x_i^{(k)}\}_k$ is monotonically decreasing and is bounded below by zero, thus it is convergent. Suppose $\{x_i^{(k)}\}_k$ converges to $\widetilde{x}_i$, and let $\widetilde{\mathbf{x}} = (\widetilde{x}_i : i = 1, \cdots, n)$. Then $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i.*

Proof: See Appendix B. ∎

Remark: In our simulations, the Iterative Descent Algorithm converges fast. For a three-user system and a target BEP $\epsilon = 10^{-3}$, it converges in fewer than 30 steps to an accuracy of six significant figures.

Theorem 2. *When $\epsilon \leq \epsilon_0$, the optimal solution $\mathbf{x}^*$ of the power control problem stated in (26) satisfies the inequality constraint with equality, i.e., $\lambda_i(\mathbf{x}^*) = \epsilon$ for $i = 1, \ldots, n$.*

Proof: For any feasible solution $\mathbf{x}$ with $\lambda_i(\mathbf{x}) < \epsilon$ for some i, applying the Iterative Descent Algorithm with $\mathbf{x}^{(0)} = \mathbf{x}$, by Lemma 3, we obtain a monotonically decreasing sequence $\{x_i^{(k)}\}_k$ which convergences to $\widetilde{x}_i$ for all i and $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i. Hence, $\sum_i \widetilde{x}_i^2 \leq \sum_i x_i^2$. Therefore, the optimal solution must satisfy the inequality constraint with equality. ∎

Define

$$\mathcal{F}_\epsilon = \{\mathbf{x} : \lambda_i(\mathbf{x}) = \epsilon, \text{for all } i\}. \tag{27}$$

The optimization problem (26) is equivalent to $\min \sum_i x_i^2$ for $\mathbf{x} \in \mathcal{F}_\epsilon$. Since λ is continuous, $\mathcal{F}_\epsilon$ is closed. Further, as $\mathcal{F}_\epsilon$ is bounded, it is compact. By Weierstrass Theorem, the optimal solution is attained.

C. Optimality of the Iterative Descent Algorithm

In this section, we want to show that when $\epsilon \leq \epsilon_0$, the function λ is injective on $\mathcal{X}_\epsilon$. Hence there is one element in $\mathcal{F}_\epsilon$ and thus the Iterative Descent Algorithm converges to the globally optimal solution.

Lemma 4. *Consider an n-user wireless communication system with channel gain matrix $G = (g_{ij})$ being row diagonally dominant. Let I be a nonempty subset of $\{1, 2, \ldots, n\}$. For two power vectors $\mathbf{x}^0$ and $\hat{\mathbf{x}}$ with $x_i^0 < \hat{x}_i$ for $i \in I$ and $x_i^0 = \hat{x}_i$ for $i \notin I$, there exists $l \in I$ s.t. $\lambda_l(\hat{\mathbf{x}}) < \lambda_l(\mathbf{x}^0)$.*

Proof: See Appendix C. ∎

Lemma 5. *For an n-user wireless communication system, if the channel gain matrix G is row diagonally dominant, the BEP function λ is injective on $\mathcal{X}_\epsilon$ for $\epsilon \leq \epsilon_0$.*

Proof: See Appendix D. ∎

In fact, using Lemma 5 and the fourth equivalent definition of M-matrix, we can prove the following theorem.

Theorem 3. *For an n-user wireless communication system, if the character matrix is an M-matrix, the BEP function λ is injective on $\mathcal{X}_\epsilon$ ($\epsilon \leq \epsilon_0$). Hence $\lambda_i(\mathbf{x}) = \epsilon$ for all i has a unique solution.*

Proof: See Appendix E. ∎

Remark: First, for the simplicity of analysis, the target BEP ϵ in (21) is set to be the same for all users. However, the case can be generalized to each user with different ϵ_i, and all the conclusions in this section hold with a slight modification. Second, if the objective function in (21) is changed to any function $f(\mathbf{x})$ such that $f(\mathbf{x}) \geq f(\mathbf{x}')$ when $\mathbf{x} \geq \mathbf{x}'$, the conclusions in this section also hold. This property of the objective function is applied in the proof of Theorem 2.

IV. Simulation Results

In this section, we compare the power control results under the Gaussian model and the non-Gaussian model. The channel gain g_{ii} is set to one for all i and g_{ij} is drawn from a uniform distribution on the interval $[0, 1]$ for all $j \neq i$. Moreover, they satisfy that the character matrix C is an M-matrix. The power spectral density N_0 of the AWGN is 10^{-10} W/Hz. The basic transmission bit rate $R = 1$Mb/s.

Briefly recall that in the Gaussian model, using BPSK modulation scheme, the BEP of user i is [7]

$$\lambda_i = Q(\sqrt{2\gamma_i}). \tag{28}$$

Therein, γ_i is the SINR of user i defined as

$$\gamma_i = \frac{g_{ii}^2 x_i^2}{\sum_{j \neq i} g_{ij}^2 x_j^2 + \eta_i}, \tag{29}$$

where η_i is the power of received thermal noise at rec$_i$.

In order to allow an equal-footing comparison, set $\eta_i = N_0 R_i$ for all i. It is obtained from

$$Q\left(\sqrt{2\frac{g_{ii}^2 x_i^2}{\eta_i}}\right) = Q\left(\frac{T_i g_{ii} x_i}{\sqrt{N_0 T_i / 2}}\right). \tag{30}$$

That is, without mutual user interference, the BEP's calculated under the Gaussian model are the same as those under the non-Gaussian model. From another point of view, such setting of η_i takes into account the bandwidth of the filter at rec_i which is related to the transmission rate R_i. It should be mentioned that most simulations in existing research of power control problem using the Gaussian model assume that η_i is a constant for all rec_i, which in fact is not always the case.

First, we consider a system involving three active users and each transmitter transmits at the same bit rate R. Fig. 2 shows the power as a function of target BEP ϵ for each user. The powers under the non-Gaussian model are found by the Iterative Descent Algorithm. The solutions under the Gaussian model are solved by $(I - \gamma H)^{-1}\gamma\eta$ [8], where I is the 3-by-3 identity matrix, $H = (g_{ij}^2)$ with diagonal element being zero, $\eta = [\eta_1, \eta_2, \eta_3]^\top$ is the thermal noise vector and $\gamma = diag(\gamma_1, \gamma_2, \gamma_3)$ with γ_i being the SINR corresponding to target ϵ by (28). It is seen that the powers obtained in the Gaussian model deviate from those in the non-Gaussian model. Roughly, the smaller the target ϵ, the lager the differences. In Fig. 3 the deviation is further evaluated in terms of the normalized square error (NSE), which is defined as

$$NSE(\epsilon) = ||\mathbf{x}^2 - \bar{\mathbf{x}}^2|| / ||\bar{\mathbf{x}}^2||. \tag{31}$$

where $\bar{\mathbf{x}}^2$ is power under the non-Gaussian model, and $\mathbf{x}^2$ is the power under the Gaussian model. These two figures show that the Gaussian model gives inaccurate results.

Next, we investigate a two-user system where the transmitters use different bit rates. For the convenience of illustration, let $g_{11} = g_{22}$ and $g_{12} = g_{21}$. The target BEP is 10^{-4}. In Fig. 4, we fix the bit rate of tran_1 to be R, while change the bit rate of tran_2 from R to $R/10$. It is observed that both transmitters decrease power as tran_2 decreases the bit rate. The decreasing power of tran_2 is much more significant than that of tran_1. It should be pointed out again that if η_i is constant as aforementioned, the powers of two transmitters will be constant in the Gaussian model. But now, η_i is set to be $N_0 R_i$, so the Gaussian model has similar results with the non-Gaussian model. However, the Gaussian model still fails to reflect the effect of bandwidth on the mutual user interference with different data rates. Fig. 5 plots the NSE curve. It can be seen that the gap between the Gaussian model and the non-Gaussian model increases with the bit rate ratio (R_1/R_2).

In Fig. 6, the bit rate of tran_1 is still fixed to R, but the bit rate of tran_2 changes from R to $10R$. We see that the powers increase with the rate ratio (R_2/R_1) for both transmitters and tran_2 has a lager variation. Combined with Fig. 4, it can be seen that the difference between the Gaussian model and the non-Gaussian model is more significant for the user with lower transmission bit rate, say user 2 in Fig. 4 and user 1 in Fig. 6.

V. Concluding Remarks

In this paper, we investigate the power control problem based on a non-Gaussian interference model. As demonstrated in the simulations, the results under the non-Gaussian model are significantly different from those under the traditional Gaussian model. We note that the analysis in this study, although specific to certain assumptions, can be extended to more general situations. For example the channels are time-varying. However, the results reported in the paper aim to shed further lights on more general system models and the corresponding power control problems based on those models.

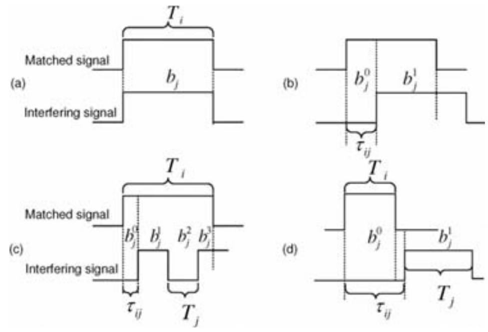

Fig. 1. Typical cases of interfering signal with different bit duration T_j in the integration interval T_i. τ_{ij} is the relative time offset and b_j is the information bit.

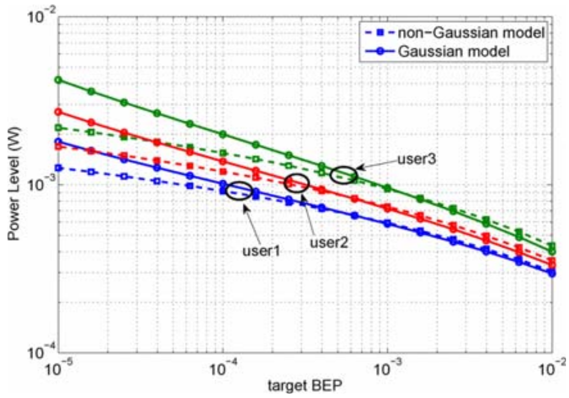

Fig. 2. Power as a function of ϵ for each of the three users. The dash lines show non-Gaussian model and the solid lines show Gaussian model.

Appendix A
Proof of Lemma 1

Fix a user i. By the property of Q-function, λ_i is strictly and monotonically decreasing with x_i.

Let $q_{ij} = g_{ij} \int_0^{T_i} a_j(t - \tau_{ij}) dt$ and $\sigma = \sqrt{N_0 T_i / 2}$. From (8),

$$\lambda_i = \mathrm{E}_{\substack{q_{ij} \\ j \neq i}} \left[Q\left(\frac{\sum_j q_{ij} x_j}{\sigma} \right) \right] \tag{32}$$

$$= \mathrm{E}_{\substack{q_{ij} \\ j \neq i, j \neq k}} \mathrm{E}_{q_{ik}} \left[Q\left(\frac{\sum_{j \neq k} q_{ij} x_j + q_{ik} x_k}{\sigma} \right) \right]. \tag{33}$$

Follows from the observation that q_{ij} has symmetrical

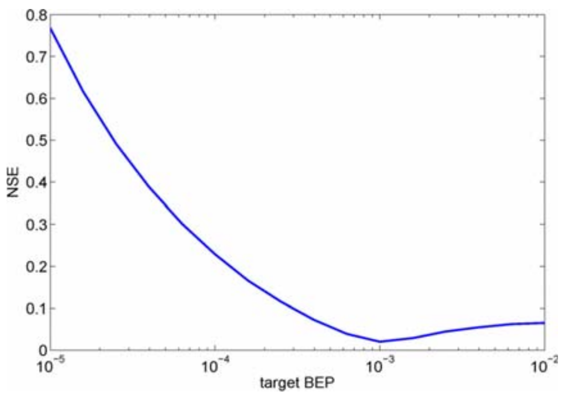

Fig. 3. NSE as a function of ϵ for non-Gaussian model and Gaussian model.

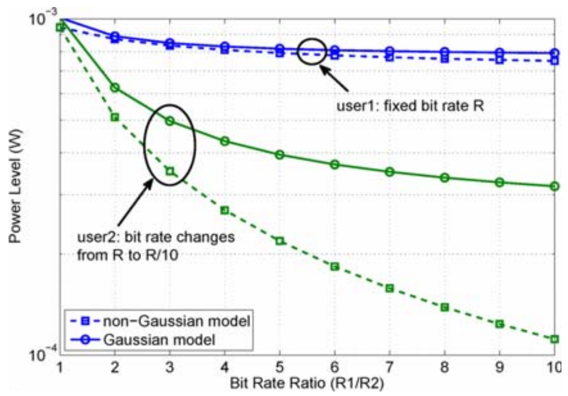

Fig. 4. Power as a function of transmission bit rate ratio (R1/R2) for non-Gaussian model and Gaussian model

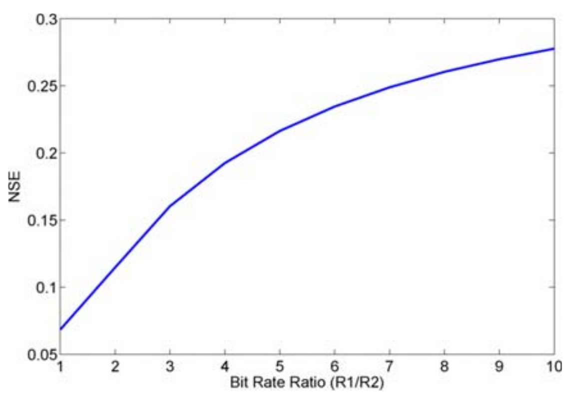

Fig. 5. NSE as a function of transmission bit rate ratio (R1/R2) for non-Gaussian model and Gaussian model.

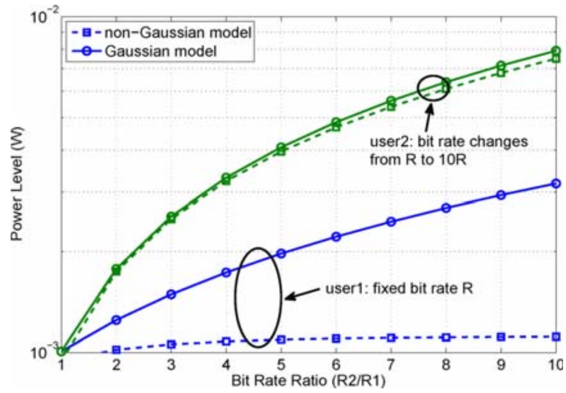

Fig. 6. Power as a function of transmission bit rate ratio (R2/R1) for non-Gaussian model and Gaussian model

distribution and define $q'_{ik} \triangleq |q_{ik}|$. We have

$$\lambda_i = \mathrm{E}_{\substack{q_{ij} \\ j\neq i, j\neq k}} \mathrm{E}_{q'_{ik}} \left[\frac{1}{2} Q \left(\frac{\sum_{j\neq k} q_{ij}x_j + q'_{ik}x_k}{\sigma} \right) + \frac{1}{2} Q \left(\frac{\sum_{j\neq k} q_{ij}x_j - q'_{ik}x_k}{\sigma} \right) \right] \quad (34)$$

$$= \mathrm{E}_{\substack{q_{ij} \\ j\neq i, j\neq k}} \mathrm{E}_{q'_{ik}} \left[\frac{1}{2} f(x_k; q_{ij}) \right]. \quad (35)$$

Since for any q_{ij}, $\frac{\partial f(x_k;q_{ij})}{\partial x_k}$ is well-defined, the differentiation can move into the expectation. Then

$$\frac{\partial \lambda_i}{\partial x_k} = \mathrm{E}_{\substack{q_{ij} \\ j\neq i, j\neq k}} \mathrm{E}_{q'_{ik}} \left[\frac{1}{2} \frac{\partial f(x_k; q_{ij})}{\partial x_k} \right] \quad (36)$$

$$= \mathrm{E}_{\substack{q_{ij} \\ j\neq i, j\neq k}} \mathrm{E}_{q'_{ik}} \left[\frac{1}{2\sigma\sqrt{2\pi}} q'_{ik} \left(-\exp\left(-\frac{(\sum_{j\neq k} q_{ij}x_j + q'_{ik}x_k)^2}{2\sigma^2} \right) + \exp\left(-\frac{(\sum_{j\neq k} q_{ij}x_j - q'_{ik}x_k)^2}{2\sigma^2} \right) \right) \right] \quad (37)$$

$$> 0. \quad (38)$$

The last inequality holds since for any $\mathbf{x} \in \mathcal{X}_C$, any q_{ij} and q'_{ik},

$$\sum_{j\neq k} q_{ij}x_j + q'_{ik}x_k \geq \sum_{j\neq k} q_{ij}x_j - q'_{ik}x_k \quad (39)$$

$$\geq T_i g_{ii} x_i - T_i \sum_{j\neq i} g_{ij}x_j \quad (40)$$

$$> 0. \quad (41)$$

Therefore, λ_i is strictly and monotonically increasing with x_k for $k \neq i$.

Appendix B
Proof of Lemma 3

The input is $\mathbf{x}^{(0)} \in \mathcal{X}_\epsilon$. When $\epsilon \leq \epsilon_0$, $\mathbf{x}^{(0)} \in \mathcal{X}_C$ (ref. Prop. 1).

If $\lambda_i(\mathbf{x}^{(0)}) = \epsilon$ for all i, then we are done. Otherwise $\lambda_i(\mathbf{x}^{(0)}) < \epsilon$ for some i. Suppose at the kth step, $\mathbf{x}^{(k)}$ is feasible (i.e., $\mathbf{x}^{(k)} \in \mathcal{X}_\epsilon \subset \mathcal{X}_C$) and $x_i^{(k)}$ is updated. In order for $\lambda_i^{(k+1)} = \epsilon \geq \lambda_i^{(k)}$, by Lemma 1, $x_i^{(k+1)} \leq x_i^{(k)}$. Consequently, $\lambda_j^{(k+1)} < \lambda_j^{(k)} \leq \epsilon$ for $j \neq i$, which implies $\mathbf{x}^{(k+1)}$ is feasible and $\mathbf{x}^{(k+1)} \leq \mathbf{x}^{(k)}$. Further $x_i^{(k+1)} > 0$ since otherwise if $x_i^{(k+1)} = 0$, $g_{ii}x_i^{(k+1)} = 0 < \sum_{j\neq i} g_{ij}x_j^{(k+1)}$, i.e., $\mathbf{x}^{(k+1)} \notin \mathcal{X}_C$, a contradiction to that $\mathbf{x}^{(k+1)}$ is feasible. Therefore, the sequence $\{x_i^{(k)}\}_k$ is monotonically decreasing and is bounded below by zero.

Let $\widetilde{\mathbf{x}} = \lim_{k\to\infty} \mathbf{x}^{(k)}$. For any arbitrarily small $\delta > 0$, since $\lambda(\mathbf{x})$ are continuous functions, there exists a sufficiently large K, when $k > K$ we have

$$\left|\lambda_i(\widetilde{\mathbf{x}}) - \lambda_i(\mathbf{x}^{(k)})\right| < \delta \qquad \forall i = 1, \ldots, n. \tag{42}$$

By the algorithm, $\lambda_i(\mathbf{x}^{(k')}) = \epsilon$ for some $k' > K$. Therefore, from (42), we have

$$|\lambda_i(\widetilde{\mathbf{x}}) - \epsilon| < \delta \qquad \forall i = 1, \ldots, n. \tag{43}$$

That is, $\lambda_i(\widetilde{\mathbf{x}}) = \epsilon$ for all i.

APPENDIX C
PROOF OF LEMMA 4

The Jacobian matrix J_λ of λ is

$$J_\lambda = \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_1} & \frac{\partial \lambda_1}{\partial x_2} & \cdots & \frac{\partial \lambda_1}{\partial x_n} \\ \frac{\partial \lambda_2}{\partial x_1} & \frac{\partial \lambda_2}{\partial x_2} & \cdots & \frac{\partial \lambda_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \lambda_n}{\partial x_1} & \frac{\partial \lambda_n}{\partial x_2} & \cdots & \frac{\partial \lambda_n}{\partial x_n} \end{bmatrix}. \tag{44}$$

By Taylor's Theorem

$$\lambda(\mathbf{x}) = \lambda(\mathbf{x}^0) + J_\lambda(\mathbf{x}^0)(\mathbf{x} - \mathbf{x}^0) + o(||\mathbf{x} - \mathbf{x}^0||). \tag{45}$$

Let $\Delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x}^0$. It is easy to see $\Delta x_i \geq 0$ for all i. Suppose $l = \arg\max_i\{\Delta x_i\}$. Obviously, $l \in I$. Let $t = 1/m$ where m is a positive integer.

Define $\mathbf{x}^k = \mathbf{x}^{k-1} + t\Delta\mathbf{x}$ for $k = 1, \ldots, m$, then $\hat{\mathbf{x}} = \mathbf{x}^m$. From (45), we have

$$\lambda(\mathbf{x}^1) = \lambda(\mathbf{x}^0) + J_\lambda(\mathbf{x}^0)t\Delta\mathbf{x} + o(t||\Delta\mathbf{x}||) \tag{46}$$

$$\lambda(\mathbf{x}^2) = \lambda(\mathbf{x}^1) + J_\lambda(\mathbf{x}^1)t\Delta\mathbf{x} + o(t||\Delta\mathbf{x}||) \tag{47}$$

$$\vdots$$

$$\lambda(\mathbf{x}^m) = \lambda(\mathbf{x}^{m-1}) + J_\lambda(\mathbf{x}^{m-1})t\Delta\mathbf{x} + o(t||\Delta\mathbf{x}||). \tag{48}$$

Summing above, we have

$$\lambda(\mathbf{x}^m) = \lambda(\mathbf{x}^0) + t\sum_{k=0}^{m-1} J_\lambda(\mathbf{x}^k)\Delta\mathbf{x} + \sum o(t||\Delta\mathbf{x}||). \tag{49}$$

We know

$$\frac{\partial \lambda_l}{\partial x_k} = \mathrm{E}_{\substack{q_{lj}\\ j\neq l}}\left[\frac{-q_{lk}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\sum_j q_{lj}x_j)^2}{2\sigma^2}\right)\right] \tag{50}$$

where $q_{ll} = T_l g_{ll}$ and $q_{lj} = g_{lj}\int_0^{T_l} a_j(t - \tau_{lj})\mathrm{d}t \geq -g_{lj}T_l$.

For any $k = 0, \cdots, m-1$, the lth component of $J_\lambda(\mathbf{x}^k)\Delta\mathbf{x}$ is

$$(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l \tag{51}$$

$$= \sum_{j=1}^{n} \frac{\partial \lambda_l}{\partial x_j}\Delta x_j \tag{52}$$

$$= \mathrm{E}_{\substack{q_{lj}\\ j\neq l}}\left[\frac{1}{\sigma\sqrt{2\pi}}(\sum_{j=1}^{n} -q_{lj}\Delta x_j)\exp\left(-\frac{(\sum_j q_{lj}x_j^k)^2}{2\sigma^2}\right)\right]. \tag{53}$$

For any q_{lj}, we have,

$$\sum_{j=1}^{n} q_{lj}\Delta x_j = q_{ll}\Delta x_l + \sum_{j\neq l} q_{lj}\Delta x_j \tag{54}$$

$$\geq g_{ll}T_l\Delta x_l + \sum_{j\neq l} -g_{lj}T_l\Delta x_j \tag{55}$$

$$\geq g_{ll}T_l\Delta x_l + \sum_{j\neq l} -g_{lj}T_l\Delta x_l \tag{56}$$

$$= T_l\Delta x_l\left(g_{ll} + \sum_{j\neq l} -g_{lj}\right) \tag{57}$$

$$> 0. \tag{58}$$

The last inequality holds since the channel gain matrix $G = (g_{ij})$ is row diagonally dominant, i.e., $g_{ll} > \sum_{j\neq l} g_{lj}$.

Therefore, $(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l < 0$ for $k = 0, \cdots, m-1$.

Further, since $\lim_{t\to 0}\frac{o(t||\Delta\mathbf{x}||)_l}{t||\Delta\mathbf{x}||} = 0$ and $\frac{|(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l|}{||\Delta\mathbf{x}||} > 0$, there exists a sufficiently large M, when $m > M$, we have

$$|o(t||\Delta\mathbf{x}||)_l| < t\left|(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l\right| = -t(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l, \tag{59}$$

that is, for $k = 0, \cdots, m-1$

$$o(t||\Delta\mathbf{x}||)_l + t(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l < 0. \tag{60}$$

Hence,

$$\lambda_l(\mathbf{x}^m) - \lambda_l(\mathbf{x}^0) = t\sum_{k=0}^{m-1}(J_\lambda(\mathbf{x}^k)\Delta\mathbf{x})_l + \sum o(t||\Delta\mathbf{x}||)_l \tag{61}$$

$$< 0. \tag{62}$$

That is

$$\lambda_l(\hat{\mathbf{x}}) = \lambda_l(\mathbf{x}^m) < \lambda_l(\mathbf{x}^0). \tag{63}$$

APPENDIX D
PROOF OF LEMMA 5

For any $\mathbf{x}$ and $\mathbf{x}' \in \mathcal{X}_\epsilon$, $\mathbf{x} \neq \mathbf{x}'$, define $I = \{i : x_i < x_i'\}$. We assume $I \neq \emptyset$. (When $I = \emptyset$, we can exchange the value of $\mathbf{x}$ and $\mathbf{x}'$.) Let $\hat{\mathbf{x}}$ be $\hat{x}_i = x_i'$ for $i \in I$ and $\hat{x}_i = x_i$ for $i \notin I$. By Lemma 4, there exists $l \in I$ s.t. $\lambda_l(\hat{\mathbf{x}}) < \lambda_l(\mathbf{x}) \leq \epsilon$. Since $\lambda_l(\hat{\mathbf{x}}) < \epsilon \leq \epsilon_0$, we have $T_l g_{ll}\hat{x}_l - T_l\sum_{j\neq l} g_{lj}\hat{x}_j > 0$, for otherwise, by (20), $\lambda_l(\hat{\mathbf{x}}) > \epsilon$, which contradicts. Following the same proof of Lemma 1, since $T_l g_{ll}\hat{x}_l - T_l\sum_{j\neq l} g_{lj}\hat{x}_j > 0$, (41) holds, and thus λ_l is monotonically increasing with x_i for $i \neq l$. Further, as $\hat{x}_i \geq x_i'$ for $i \notin I$, $\lambda_l(\mathbf{x}') \leq \lambda_l(\hat{\mathbf{x}})$ and hence $\lambda_l(\mathbf{x}') < \lambda_l(\mathbf{x})$. In conclusion, $\lambda(\mathbf{x}) = \lambda(\mathbf{x}')$ if and only if $\mathbf{x} = \mathbf{x}'$, that is, the function λ is injective on $\mathcal{X}_\epsilon$.

APPENDIX E
PROOF OF THEOREM 3

Consider an n-user wireless communication system with channel gain matrix $G = (g_{ij})$ and its character matrix C is an M-matrix. If G is row diagonally dominant, By Lemma 5, we are done. Otherwise by the fourth equivalent definition of M-matrix, there exists a positive diagonal matrix $D = \text{diag}(d_1, \cdots, d_n)$, such that CD is strictly row diagonally dominant. In detail:

$$CD = \begin{bmatrix} g_{11} & -g_{12} & \cdots & -g_{1n} \\ -g_{21} & g_{22} & \cdots & -g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{n1} & -g_{n2} & \cdots & g_{nn} \end{bmatrix} \cdot \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \tag{64}$$

$$= \begin{bmatrix} d_1 g_{11} & -d_2 g_{12} & \cdots & -d_n g_{1n} \\ -d_1 g_{21} & d_2 g_{22} & \cdots & -d_n g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -d_1 g_{n1} & -d_2 g_{n2} & \cdots & d_n g_{nn} \end{bmatrix}, \tag{65}$$

and

$$|d_i g_{ii}| > \sum_{j \neq i} |-g_{ij} d_j| \quad \text{for all } i. \tag{66}$$

Hence,

$$d_i g_{ii} > \sum_{j \neq i} g_{ij} d_j \quad \text{for all } i. \tag{67}$$

Define a matrix $A = (a_{ij})$ by letting $a_{ij} = g_{ij} d_j$ for all i and all j. Then A is row diagonally dominant. By Lemma 5, under channel gain matrix A, the BEP function λ_A is injective on $\mathcal{X}_{A_\epsilon} = \{\mathbf{x} : \lambda_{A_i}(\mathbf{x}) \leq \epsilon, i = 1, \cdots, n\}$.

Observe that under channel gain matrix G, the BEP function λ_G has the relation: $\lambda_G(D\mathbf{x}) = \lambda_A(\mathbf{x})$ for any $\mathbf{x}$. Define $\mathcal{X}_G = \{D\mathbf{x} : \mathbf{x} \in \mathcal{X}_{A_\epsilon}\}$. Since λ_A is injective on $\mathcal{X}_{A_\epsilon}$, λ_G is injective on $\mathcal{X}_G$.

Moreover,

$$\begin{aligned} \mathcal{X}_{G_\epsilon} &= \{\mathbf{x} : \lambda_{G_i}(\mathbf{x}) \leq \epsilon, i = 1, \cdots, n\} && (68) \\ &= \{\mathbf{x} : \lambda_{A_i}(D^{-1}\mathbf{x}) \leq \epsilon, i = 1, \cdots, n\} && (69) \\ &= \{\mathbf{x} : D^{-1}\mathbf{x} \in X_{A_\epsilon}\} && (70) \\ &= \{D\mathbf{x} : \mathbf{x} \in X_{A_\epsilon}\} && (71) \\ &= \mathcal{X}_G && (72) \end{aligned}$$

Therefore λ_G is injective on $\mathcal{X}_{G_\epsilon}$ and the solution for $\lambda_{G_i}(\mathbf{x}) = \epsilon$ for all i is unique.

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